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CRC 293

MODELING THE OCEAN - INTRODUCTION TO WAVE PROPAGATION IN A TURBULENT MEDIUM

CENTER FOR NAVAL ANALYSES
1401 Wilson Boulevard
Arlington, Virginia 22209
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January 1976

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Prepared for:

OFFICE OF NAVAL RESEARCH
Department of the Navy
Washington, D.C. 20350

OFFICE OF THE CHIEF OF NAVAL OPERATIONS (Op96)
Department of the Navy
Washington, D.C. 20380

02 029300.00

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CRC 293	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Modeling the Ocean -- Introduction to Wave Propagation in a Turbulent Medium		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Alfred Kaufman		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0001
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Naval Analyses 1401 Wilson Boulevard Arlington, Virginia 22209		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Washington, D.C. 20350		12. REPORT DATE January 1976 13. NUMBER OF PAGES 55
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Office of the Chief of Naval Operations (Op96) Department of the Navy Washington, D.C. 20380		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This Research Contribution does not necessarily represent the opinion of the Department of the Navy.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

acoustic refraction, acoustics, KCT framework, layers, mathematical analysis, mathematical models, ocean models, sound transmission, turbulent flow, underwater sound, wave equations, wave propagation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The observation fundamental to this work is that the ocean is usually in a state of turbulent motion. Correspondingly, the value of the temperature at every point in the ocean undergoes irregular fluctuations. In particular, since the index of refraction of the ocean is a function of temperature, we shall take the viewpoint that the refractive index is random and assume that the Kolmogorov theory of locally homogeneous and isotropic turbulence provides a sufficiently good description of the refractive index microstructure. To extract information concerning the randomness

20 of an acoustic wave propagating through this turbulent and unbounded ocean, we make use of the wave equation to connect the statistical properties of the random medium to the implied statistical properties of the wave parameters within the framework of a correlation theory.

We accomplish this only to first order in perturbation theory, thus restricting the realm of validity of our results to high frequencies and small refractive index fluctuations. The structure function of the logrithmic amplitude we find, generalizes similar results of Tatarski and Chernov away from the transversal, correspondingly longitudinal restrictions inherent in their work.

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INTRODUCTION

The first thing one learns about sound in the ocean is that it fluctuates. Such variability is much more than merely conspicuous; it is quite general and ever present. Thus, variability exists in all parameters of the sonar equation such as propagation loss, target strength, ambient noise and recognition differential, though each time due to rather different causes. Fluctuations in the ocean are also relevant to understanding the performance of naval acoustic equipment in that they unmistakably, and sometimes drastically, influence such performance. Typical is the observation that targets tend to be detected at ultra-long ranges during periods of signal surges and lost at short ranges during periods of signal fades. To put it differently, the actual detection probability falls off more slowly in field exercises than would be expected in the absence of fluctuations.

Of the many kinds of variability mentioned above, fluctuations in propagation loss have received a great deal of attention. In consequence, a substantial amount of information both experimental and theoretical has been gathered concerning candidate mechanisms of transmission fluctuability. It has thus become apparent that, under certain conditions at least, sound wave scattering off the temperature microstructure of the medium can account for most of the observed variability. Hereafter, we shall refer to this mechanism and the attendant body of theory as the Kolmogorov, Chernov, and Tatarski (KCT) framework.[1]

In this paper we offer a straightforward generalization of the KCT results away from the transversal correlation restriction inherent therein. The academic value of this work is limited; the tactical value thereof is, however, paramount in that for real life employment of sonar detection the transversal correlation case is of vanishing measure.

The observation fundamental to our work is that the ocean is usually in a state of turbulent motion. Correspondingly, the values of the temperature at every point in the ocean undergo irregular fluctuations; the values of the temperature at different spatial point at the same instant in time also differ from one another in a random fashion. What has been said applies as well to all other oceanographic quantities. In particular, since the index of refraction of the ocean is a function of temperature and salinity, we can take the viewpoint that the refractive index field is random. To become specific, we shall assume that the Kolmogorov theory of locally homogeneous and isotropic turbulence[2] provides a sufficiently good description of the refractive index microstructure.

To extract information concerning the randomness of an acoustic wave propagating through this turbulent and unbounded ocean, we make use of the wave equation to connect the statistical properties of the random medium to the implied statistical properties of wave parameters within the framework of a correlation theory. We accomplish this only to first order in perturbation theory thus restricting the realm of validity of our results to high frequencies and small refractive index fluctuations.

The main output of this analysis is the structure functions of the logorithmic amplitude and phase fluctuations. For conciseness, we have explicitly handled only the amplitude fluctuations, but the same treatment can be continued with no difficulty to include the phase fluctuations as well.

We thus find that the correlation length of the logorithmic amplitude fluctuations is given by,

$$\rho_{corr}(\theta) \sim \begin{cases} \dfrac{\sqrt{\lambda L}}{\cos\theta} , & 0 \le \theta < \theta_0 \\ \\ \dfrac{4\pi L_0^2}{\lambda \sin\theta} , & \theta_0 < \theta \le \pi/2 \end{cases} , \quad \sqrt{\lambda L} << L_0$$

where θ measures the angle between the observation plane and the wave front, λ is the wave length, L the distance between the source and the receiver, L_0 the outer scale of turbulence, and θ_0 is the solution of

$$\frac{\sqrt{\lambda L}}{\cos\theta} = \frac{4\pi L_0^2}{\lambda \sin\theta} \quad .$$

We also find that,

$$\left\langle (\log A/A_0)^2 \right\rangle = .31\, C_n^2\, k^{7/6}\, L^{11/6}$$

$$(\sqrt{\lambda L} << L_0)$$

where C_n^2 measures the strength of the Kolmogorov "two-thirds" law for the refractive index fluctuations, k stands for the wave number $2\pi/\lambda$ and where A/A_0 represents the amplitude of the perturbed wave normalized to the plane wave amplitude A_0.

The first three chapters are intended to familiarize the reader with the mathematical language of random fields extensively used herein, with the physical picture of turbulent flow, and with the corresponding microstructure of the refractive index field respectively. The wave equation is developed in chapter D.1, and the small wavelength modification of Rytov to the first order perturbation theory appears in D.2. We solve the Rytov equation using the method of spectral expansions in chapter D.3. Explicit evaluation of the solution for the Kolmogorov model is included in chapter D.4. The analysis concludes with a chapter on comparison with experimental data[3] gathered under conditions where the turbulence mechanism is expected to dominate.

A. RANDOM FUNCTIONS

- A random function on T is a family of random variables $\{\xi(t), \xi(s), \ldots\}$ corresponding to all elements $(t, s, \ldots)$ in the set T.

- We shall regard the random function $\xi(t)$ as specified if for each subset $(t_1, t_2, \ldots, t_n)$ in T we are given the distribution function

$$F_{t_1,t_2,\ldots,t_n}(\chi_1,\chi_2,\ldots,\chi_n) \equiv P\left\{\xi(t_1)<\chi_1,\ldots,\xi(t_n)<\chi_n\right\} . \tag{A-1}$$

In most actual physical problems it is excessively complicated to measure these distribution functions. They are also too cumbersome to be used in practice. It is, therefore, convenient to restrict oneself to the simplest numerical characteristics of the multidimensional distribution, the moments

$$\mu_{m_1,m_2,\ldots,m_n}(t_1,t_2,\ldots,t_n) \equiv \langle \xi^{m_1}(t_1)\, \xi^{m_2}(t_2) \ldots \xi^{m_n}(t_n)\rangle . \tag{A-2}$$

Here,

$$\langle \xi^{m_1}(t_1)\, \xi^{m_2}(t_2) \ldots \xi^{m_n}(t_n)\rangle \equiv \tag{A-3}$$

$$\equiv \int_{-\infty}^{\infty} \chi_1^{m_1} \chi_2^{m_2} \cdots \chi_n^{m_n}\, dF_{t_1,t_2,\ldots,t_n}(\chi_1,\chi_2,\ldots,\chi_n)$$

where the integration on the right-hand side is to be performed in the Stieltjes sense. If the distribution F has a density,

$$f_{t_1,t_2,\ldots,t_n}(\chi_1,\chi_2,\ldots,\chi_n) = \frac{\partial^n F_{t_1,t_2,\ldots,t_n}(\chi_1,\chi_2,\ldots,\chi_n)}{\partial\chi_1\partial\chi_2\cdots\partial\chi_n} \tag{A-4}$$

the moment reduces to the standard form:

$$\mu_{m_1, m_2, \ldots, m_n}(t_1, t_2, \ldots, t_n) = \tag{A-5}$$

$$= \int_{-\infty}^{\infty} d\chi_1 d\chi_2 \cdots d\chi_n \, \chi_1^{m_1} \chi_2^{m_2} \cdots \chi_m^{m_n} f_{t_1, t_2, \ldots, t_m}(\chi_1, \chi_2, \ldots, \chi_n).$$

We shall further restrict ourselves by taking into account only those properties of a random function which are determined by its first and second moments, the mean value and the correlation function:

$$m(t) = \langle \xi(t) \rangle \quad ; \quad B(t,s) = \langle \xi(t)\, \xi(s) \rangle \quad . \tag{A-6}$$

• The random function $\xi(t)$ is called stationary if all the finite-dimensional distribution functions defining $\xi(t)$ are invariant under translations of the whole group of points $(t_1, t_2, \ldots, t_n)$ in T,

$$F_{t_1+\tau\, t_2+\tau, \ldots, t_n+\tau}(\chi_1, \chi_2, \ldots, \chi_n) = F_{t_1, t_2, \ldots, t_n}(\chi_1, \chi_2, \ldots, \chi_n) \quad . \tag{A-7}$$

If $\xi(t)$ is stationary, then obviously:

$$m(t) = m \quad ; \quad B(t,s) = B(t-s) \quad . \tag{A-8}$$

The physical content of the concept of stationarity is clear. It means that $\xi(t)$ describes the time variation of a numerical characteristic ξ of an event such that none of the observed macroscopic factors influencing the occurrence of the event changes in time.

• To determine the mean value $m(t)$ and the correlation function $B(t, s)$ of a random function $\xi(t)$ we must first take a large number N of realizations of $\xi(t)$, written $\xi^{(1)}(t), \xi^{(2)}(t), \ldots, \xi^{(N)}(t)$ and then calculate the arithmetic mean of $\xi^{(j)}(t)$ over j for every value of t, or the arithmetic mean of $\xi^{(j)}(t)\xi^{(j)}(s)$ for every pair of values t and s. However, in practice, observation of a random function and the subsequent processing of the data usually turn out to be quite complicated. It would, therefore, be very desirable to be able to get along with as small a number of realization as possible. Indeed, the practical value of the correlation theory of stationary random functions is to a considerable extent due to the fact that if $\xi(t)$ is stationary, its mean value m and its correlation function $B(t)$ can usually be calculated by using just one realization of $\xi(t)$. This is the content of the ergodic theorem.

More precisely, if $\xi(t)$ is a stationary random function satisfying certain quite general conditions, then

$$<\xi(t)> = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \xi(t) \tag{A-9}$$

$$<\xi(t+\tau)\xi(t)> = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \xi(t+\tau)\xi(t) \tag{A-10}$$

where the integrals are defined as the limits of the corresponding approximating sums:

$$\int_0^T dt\, \xi(t) = \lim_{N\to\infty} \frac{T}{N} \sum_{n=1}^{N} \xi(n\frac{T}{N}) \tag{A-11}$$

$$\int_0^T dt\ \xi(t+\tau)\xi(t) = \lim_{N\to\infty} \frac{T}{N} \sum_{n=1}^{N} \xi(n\frac{T}{N}+\tau)\xi(n\frac{T}{N}) \quad . \tag{A-12}$$

We notice, that in the statement of the ergodic theorem, the limit is taken to mean limit in the mean-square.

- A stationary random function can be represented in the form of a stochastic Fourier-Stieltjes integral,

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\emptyset(\omega) \tag{A-13}$$

where stationarity requires:

$$<d\,\emptyset(\omega)d\,\emptyset^*(\lambda)> = \delta(\omega-\lambda)W(\omega)d\omega\, d\lambda \quad . \tag{A-14}$$

Hence:

$$B(t-s) = \int_{-\infty}^{\infty} e^{i(\omega t-\lambda s)} < d\phi(\omega) d\phi^*(\lambda)> \tag{A-15}$$

becomes,

$$B(t-s) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-s)} W(\omega) \quad . \tag{A-16}$$

Obviously, $W(\omega) \geq 0$, and therefore the Fourier transform of a correlation function must be non-negative. Khinchin showed(4) that the converse assertion is also true: if the Fourier transform of B(t) is non-negative, then there exists a stationary random function $\xi(t)$ with B(t) as its correlation function. $W(\omega)$ is called the spectral density function of $\xi(t)$.

- In general, the random functions encountered in practical applications can very often be regarded to a high degree of accuracy as stationary. However, the opposite can also occur. The mean value of some oceanic or meteorological variable undergoes comparatively slow and smooth changes, and hence they are random functions that cannot be regarded as stationary.

In the case where $\xi(t)$ represents a non-stationary random function, i.e., where $<\xi(t)>$ changes in the course of time, we can consider instead of $\xi(t)$ the difference,

$$\Delta_\tau(t) \equiv \xi(t+\tau) - \xi(t) \quad . \tag{A-17}$$

For values of τ which are not too large, slow changes in the function $\xi(t)$ do not affect the value of this difference, and it can be regarded, at least approximately, as a stationary random function:

$$<\Delta_\tau(t)> \simeq 0 \quad . \tag{A-18}$$

Random functions that possess this property are called random functions with stationary increments (Kolmogorov).(2) It is easy to show that the correlation function

$$B_\Delta(t,s) = <\Delta_\tau(t)\Delta_\tau(s)> \tag{A-19}$$

is expressible as a linear combination of the structure functions:

$$D(t,s) = \left\langle \left(\xi(t) - \xi(s) \right)^2 \right\rangle \quad . \tag{A-20}$$

For $B_\Delta(t,s)$ to depend only on the difference $(t-s)$, it shall suffice to have,

$$D(\tau) = \left\langle \left(\xi(t+\tau) - \xi(t) \right)^2 \right\rangle \quad . \tag{A-21}$$

Roughly speaking, the value of $D(\tau)$ characterizes the intensity of those fluctuations of $\xi(t)$ with periods which are smaller than or comparable with τ .

- A random function with stationary increments can be represented in the form,

$$\xi(t) = \xi(0) + \int_{-\infty}^{\infty} (1 - e^{i\omega t}) d\phi(\omega) \tag{A-22}$$

where $\xi(0)$ is a random variable, and the Fourier-Stieltjes amplitudes $d\phi(\omega)$ obey the condition

$$\langle d\phi(\omega) d\phi^*(\lambda) \rangle = \delta(\omega - \lambda) W(\omega) d\omega d\lambda \quad . \tag{A-23}$$

Hence,

$$D(t) = 2 \int_{-\infty}^{\infty} d\omega \, (1 - \cos \omega t) W(\omega) \tag{A-24}$$

and

$$W(\omega) \geq 0 \quad . \tag{A-25}$$

B. THE MICROSTRUCTURE OF TURBULENT FLOW

For us, the most important fact about the ocean is that it is usually in a state of turbulent motion. We shall therefore need some basic information concerning the statistical properties of developed turbulent flow. The statistical theory of turbulence, was initiated in the papers of Friedmann & Keller. A very important advance was achieved in 1941, when Komogorov,[2] Obukhov[5] and later Onsager, von Weisacker and Heisenberg established the laws which characterize the basic properties of the microstructure of turbulent flow at very large Reynolds numbers.

Consider an initially laminar flow of a viscous fluid. This flow can be characterized by the values of the kinematic viscosity ν , the characteristic velocity scale v and the characteristic length L . The quantity L specifies the dimensions of the flow as a whole and arises from the boundary conditions of the fluid dynamics problem. Suppose now, that for some reason or other, a velocity fluctuation v'_Λ occurs in a region of size Λ of the basic laminar flow. The characteristic time $\tau_\Lambda = \Lambda / v'_\Lambda$ which corresponds to this fluctuation specifies the order of magnitude of the time required for the occurrence of the fluctuation. The energy per unit mass of the given fluctuation is given by v'^2_Λ . Thus, when the velocity fluctuation under consideration occurs, the amount of energy per unit time per unit mass which goes over from the initial flow to the fluctuational motion is equal in order of magnitude to,

$$v'^2_\Lambda / \tau_\Lambda \simeq v'^3_\Lambda / \Lambda \quad .$$

On the other hand, the local velocity gradients thus developed are given by the ratio v'_Λ / Λ , and therefore the energy dissipated as heat per unit mass of the fluid per unit time is of order of magnitude

$$\epsilon = \nu v'^2_\Lambda / \Lambda^2 \quad .$$

If the velocity fluctuation which arises is to have existence of its own, it is clearly necessary that the inequality

$$v'^3_\Lambda / \Lambda > \nu v'^2_\Lambda / \Lambda^2 \tag{B-1}$$

hold, i.e., that

$$Re\Lambda \equiv \Lambda v'_\Lambda / \nu > 1 \quad . \tag{B-2}$$

Since all these calculations are accurate only to within undetermined numerical factors, it would be more appropriate to write instead of equation B-2.

$$Re_\Lambda > Re_{cr}$$

Where Re_Λ denotes the "inner" Reynolds number corresponding to fluctuations of size Λ , and Re_{cr} is some fixed number that cannot be determined precisely. Notice, however, that if

$$Re_{\Lambda} > Re_{cr} \quad ,$$

the direct energy dissipation for the velocity fluctuation with size Λ is small compared to the energy it receives and thus the fluctuation can transfer almost all its energy to smaller perturbations. Consequently, the quantity v'^{3}_{Λ}/Λ, which represents the energy per unit mass received per unit time by eddies of nth order from eddies of (n-1)'th order and thereupon transferred to eddies of (n+1)th order, is a constant for perturbations of almost all sizes. In the smallest velocity perturbation with size Λ_0, this energy is converted into heat. The rate of dissipation of energy into heat is of the order $\epsilon \sim \nu v_0^{\,2}/\Lambda_0^{\,2}$, and hence for velocity fluctuations of all scales we have

$$\epsilon \sim v'^{3}_{\Lambda}/\Lambda \tag{B-3}$$

for the energy transmitted down the eddies ordered chain. It is naturally clear that if the condition $VL/\nu > Re_{cr}$ is not met for the flow as a whole, the laminar motion is stable.

The largest eddies which arise as a result of the instability of the basic flow are of course not isotropic, since they are influenced by the special geometric properties of the flow. However, these special properties no longer influence the eddies of sufficiently high order, and therefore are good grounds for considering the latter to be isotropic. Then, since eddies with dimensions much larger than $|\vec{\rho}|$ do not influence the two point function $\left[\vec{v}\,(\vec{r} + \vec{\rho}) - \vec{v}\,(\vec{r})\right]$, this difference will depend only on isotropic eddies.

We thus arrive naturally at the scheme of a locally isotropic random field. The random field is hence uniquely characterized by the longitudal structure function $D_{\parallel}(\rho)$. The form of the structure function can be established by using the qualitative consideration developed above. In fact, consider a value of ρ that is large compared to the inner scale Λ_0 of the turbulence and small compared with the outer scale L_0 of turbulence. Then the velocity difference at $\vec{r}$ and $\vec{r} + \vec{\rho}$ is mainly due to eddies with dimension comparable to ρ. But the only parameter which characterizes such eddies is the energy dissipation rate ϵ. Thus, we can assert that $D_{\parallel}(\rho)$ is only a function of ρ and ϵ. Dimensional arguments then provide

$$\boxed{D_{\parallel}(\rho) = C(\epsilon\rho)^{2/3}} \tag{B-4}$$

$$\Lambda_0 \ll \rho \ll L_0$$

where C is a dimensionless constant of order unity. This equation represents the famous Kolmogorov "two-thirds law." For $\rho \ll \Lambda_0$, changes of velocity occur smoothly, since now the relative motions are laminar. The velocity difference can therefore be expanded in power series of ρ, and

$$\boxed{D_{\parallel}(\rho) = a\,\rho^2} \qquad \text{(B-5)}$$

$$\rho \ll \Lambda_0$$

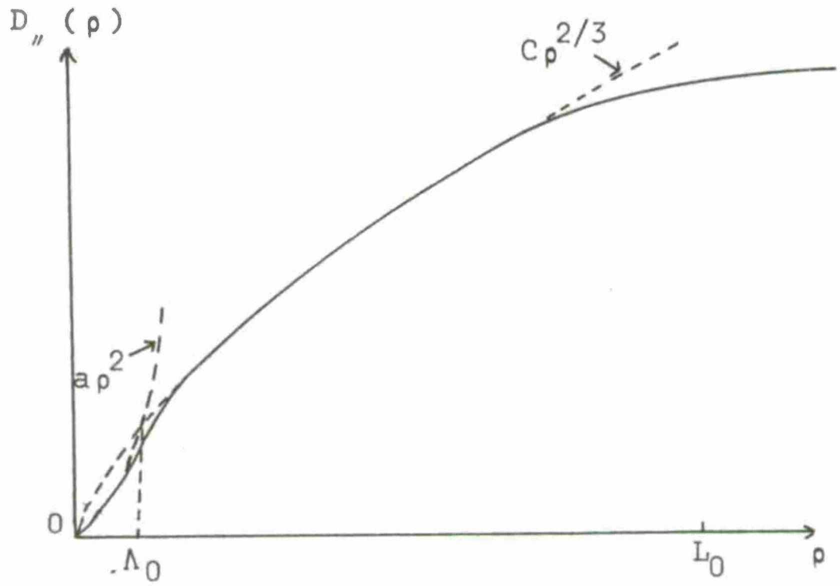

FIG. 1. GENERAL SHAPE OF THE STRUCTURE FUNCTION $D_{\parallel}(\rho)$

In equation (B-4) it is assumed that $\rho \ll L_0$, where L_0 is the outer scale of the turbulence. When ρ is increased, the condition $\rho \ll L_0$ is violated. Then the large eddies, which cannot be regarded as isotropic and homogeneous, begin to influence the value of $\vec{v}(\vec{r}+\vec{\rho}) - \vec{v}(\vec{r})$. In this case, the structure function $D_{\parallel}(\rho)$ depends on the coordinates of both observation points, and no universal law can be given which describes the structure function for large values of ρ. We can only state that the growth of the structure function slows down for $\rho \gg L_0$. Figure 1 shows the general shape of the longitudinal structure function $D_{\parallel}(\rho)$. For small values of ρ, the curve can be replaced by a parabola with great accuracy, then the part of the curve corresponding to the "two-thirds" law begins, and finally, in the region of the outer scale of turbulence, the curve starts to saturate.

In many applications, it is often expedient to regard the values of ρ as not being bounded from above by the value of L_0. However, if we cannot neglect the saturation of the structure function, it is necessary to use interpolation formulas which approximately describe the behavior of the structure function for large values of ρ. For small values of ρ, these formulas must reduce to the same values of $D_{/\!/}(\rho)$ as given in equation (B-5). In what follows we shall use the Karman function,

$$D_{/\!/}(\rho) = \frac{2}{3} \langle v'^2 \rangle \left[1 - \frac{2^{2/3}}{\Gamma(1/3)} \left(\frac{\rho}{L_0}\right)^{1/3} K_{1/3}\left(\frac{\rho}{L_0}\right) \right] . \tag{B-6}$$

Here $\langle v'^2 \rangle$ is the mean square velocity fluctuation, L_0 the outer scale of turbulence and $K_{1/3}\left(\frac{\rho}{L_0}\right)$ is the 1/3 order Bessel function of the second kind of imaginary argument.

For $\frac{\rho}{L_0} << 1$ the Karman structure function is approximately equal to

$$D_{/\!/}(\rho) \sim \frac{\sqrt{\pi}}{3\Gamma(7/6)} \langle v'^2 \rangle \left(\frac{\rho}{L_0}\right)^{2/3} , \tag{B-7}$$

and hence coincides for $\rho << L_0$ with the Kolmogorov "two-thirds" law.

C. TURBULENT MIXING OF PASSIVE ADDITIVES

• Although the foregoing discussion shall prove quite essential, let us observe that it is really the structure of the temperature and salinity fields that ultimately determine the characteristics of wave propagation in the free ocean, in that the wave velocity C is a function of temperature, salinity and depth. We shall now take the view that the temperature and salinity fields can be regarded as conservative, passive additives to the turbulent ocean. If a volume of liquid is characterized by a concentration θ of additive, then by saying that the additive is conservative we mean that the quantity θ does not change when the volume element is shifted about in space. By the additive being passitive is meant that the quantity θ does not affect the dynamical regime of the underlying turbulence.

As usual, we separate the value of the quantity θ into the mean value $\langle \theta \rangle$ and the fluctuation θ',

$$\theta = \langle \theta \rangle + \theta'$$

and define the following measure of the inhomogeniety of the spatial distribution of θ,

$$G = \frac{1}{2} \int_V dV < \theta'^2 > \tag{C-1}$$

so that G vanishes if, and only if, θ' vanishes identically in V . Using the equation of molecular diffusion which must be obeyed by the concentration θ of the passive additive we can easily show that,

$$\frac{\partial G}{\partial t} = \int_V dV \left[K(\nabla < \theta >)^2 - D < (\nabla \theta')^2 > \right] \tag{C-2}$$

where D represents the coefficient of molecular diffusion and K the coefficient of turbulent diffusion, with $K >> D$. For clarity, we recall that if $\vec{q}_M$ represents the mean flow of θ caused by molecular diffusion,

$$\vec{q}_M = - D\nabla < \theta >$$

and, similarly, if $\vec{q}_T$ represents the density of turbulent flow of θ ,

$$\vec{q}_T = - K\nabla < \theta > \quad .$$

Thus, the time dependence of G is governed by the delicate interplay of two distinct physical processes, turbulent mixing and molecular diffusion. As a result of turbulent mixing the inhomogeneity of the spacial distribution of θ is increased and large local gradients of θ are created. Only after these large gradients have appeared does the process of molecular diffusion play a significant role by smoothing out the spatial distribution of θ .

The quantity,

$$N \equiv D < (\nabla \theta')^2 > \tag{C-3}$$

represents the amount of inhomogeneity which disappears per unit time due to molecular diffusion and it is analogous to the energy dissipation rate ϵ . Correspondingly, the quantity $K(\nabla < \theta >)^2$ represents the amount of inhomogeneity which appears per unit time due to the turbulence, and is similar to v_Λ^3/Λ , the rate of production of the energy of the velocity fluctuations.

We can carry out an even more detailed analogy between the velocity fluctuation in a turbulent flow and the concentration fluctuations of a passive additive θ . Concentration inhomogeneities θ'_Λ with geometrical dimensions Λ appear as a result of the action of velocity field perturbations with dimensions Λ and characteristic velocities v_Λ. The amount of inhomogeneities appearing per unit time due to turbulence is clearly given by, $v_\Lambda \theta'{}_\Lambda{}^2/\Lambda$. The rate of levelling out of the inhomogeneities $\theta'{}_\Lambda$ is of the order of $D\theta'{}_\Lambda{}^2/\Lambda^2$. When

$$v_\Lambda \theta'{}_\Lambda{}^2/\Lambda \gg D\theta'{}_\Lambda{}^2/\Lambda^2 \tag{C-4}$$

that is, when

$$\Lambda v_\Lambda/D \gg 1 \quad , \tag{C-5}$$

the inhomogeneity $\theta'{}_\Lambda$ is not dissipated by the action of molecular diffusion, but rather has a stable existence and can subsequently subdivide into smaller eddies. This process of subdivision proceeds until inhomogeneities appear for which

$$\Lambda v_\Lambda/D \sim 1 \quad . \tag{C-6}$$

These inhomogeneities are dissipated by the process of molecular diffusion at a rate equal to N.

Thus, the amount of inhomogeneity transferred per unit time from the largest eddies down the chain to the smallest eddies is constant and equal to the rate N at which the inhomogeneity is dissipated. Again, the largest inhomogeneities in the distribution of θ are not isotropic. However, the smallest inhomogeneities can be considered isotropic. Hence, $\theta(\vec{r}+\vec{\rho}) - \theta(\vec{r})$, determined mainly by inhomogeneities of size ρ, can be considered statistically isotropic for values of $\rho \ll L_0$. It follows, then, that $\theta(\vec{r})$ can be regarded as a locally isotropic random field. Correspondingly, the structure function $D(\rho)$ can depend only on ρ, N, ϵ. Dimensional considerations lead to,

$$\boxed{D(\rho) = C_\theta^{\,2}\ \rho^{2/3}} \tag{C-7}$$

$$\Lambda_0 \ll \rho \ll L_0$$

which is the, by now familiar, "two-thirds" law for the concentration of a passive additive (Obukhov).(6)

D. PARAMETER FLUCTUATIONS OF ACOUSTIC WAVES PROPAGATING IN A TURBULENT OCEAN AT SMALL WAVELENGTHS

1. The Wave Equation for Acoustic Propagation

Consider a fluid medium in a state of equilibrium. We shall denote the equilibrium density by ρ_0 and the equilibrium pressure by p_0. The actual density at the point $\vec{r}$ and at time t shall be denoted by $\rho(\vec{r},t)$. Hence, the relative change in density due to the passing of the acoustic wave will be given by

$$\hat{\rho}(\vec{r},t) = \rho(\vec{r},t) - \rho_0 \quad .$$

Similarly the difference between the actual pressure and the equilibrium value, will be denoted by,

$$\hat{p}(\vec{r},t) = p(\vec{r},t) - p_0 \quad .$$

We must now find an equation giving the dependence of $\hat{p}$ on $\vec{r}$ and t. To accomplish this, we observe that the total pressure p and density ρ satisfy,

$$\rho \frac{d\vec{v}}{dt} = -\nabla p \qquad \text{(Newton's second law)} \qquad \text{(D-1)}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v}) = 0 \qquad \text{(continuity equation)} \qquad \text{(D-2)}$$

where $\vec{v}$ represents the velocity of the acoustic oscillations. Discarding terms of second and higher orders of smallness in the small quantities $\hat{p}$, $\hat{\rho}$, and $\vec{v}$, one has:

$$\rho_0 \frac{\partial\vec{v}}{\partial t} = -\nabla\hat{p} \qquad \text{(D-3)}$$

$$\frac{\partial\hat{\rho}}{\partial t} + \rho_0 \nabla\vec{v} = 0 \quad . \qquad \text{(D-4)}$$

Differentiating with respect to time the continuity equation D-4

$$\frac{\partial^2\hat{\rho}}{\partial t^2} + \rho_0 \nabla\left(\frac{\partial\vec{v}}{\partial t}\right) = 0 \quad . \qquad \text{(D-5)}$$

We can now replace $\partial\vec{v}/\partial t$ in equation D-5 from Newton's equation D-3 to obtain:

$$\frac{\partial^2 \hat{\rho}}{\partial t^2} - \nabla^2 \hat{p} = 0 \quad . \tag{D-6}$$

To find one more relationship between $\hat{p}$ and $\hat{\rho}$, we must make an assumption about the thermodynamic character of the acoustic process. Specifically, we shall assume that alterations in pressure and density are so rapid that no heat energy has time to flow away from the compressed part of the fluid before this part is no longer compressed. Such compressions are said to be adiabatic. Evidently, the amount of heat changes by,

$$dQ = \left(\frac{\partial Q}{\partial V}\right)_p dV + \left(\frac{\partial Q}{\partial p}\right)_V dp \quad , \tag{D-7}$$

and introducing the specific heats

$$c_V \equiv \left(\frac{\partial Q}{\partial T}\right)_V \quad ; \quad c_p \equiv \left(\frac{\partial Q}{\partial T}\right)_p \quad ,$$

we have:

$$dQ = c_p\left(\frac{\partial T}{\partial V}\right)_p dV + c_V \left(\frac{\partial T}{\partial p}\right)_V dp \quad . \tag{D-8}$$

If the fluid is a perfect gas, $pV = RT$, and hence,

$$\frac{dT}{T} = \frac{dp}{p} + \frac{dV}{V} \quad ,$$

providing:

$$\left(\frac{\partial T}{\partial V}\right)_p = \frac{T}{V} \quad ; \quad \left(\frac{\partial T}{\partial p}\right)_V = \frac{T}{p} \quad . \tag{D-9}$$

With these relations, the heat change equation becomes:

$$dQ = T\left(c_p \frac{dV}{V} + c_V \frac{dp}{p}\right) \tag{D-10}$$

For adiabatic processes, $dQ = 0$ and hence equation D-10 provides,

$$\frac{dp}{p} = - \gamma \frac{dV}{V} \tag{D-11}$$

where

$$\gamma \equiv \frac{C_p}{C_V} \tag{D-12}$$

We rewrite equation D-11 in the form,

$$\hat{p} = \gamma \frac{p_0}{\rho_0} \hat{\rho}$$

or, rather,

$$\hat{p} = C^2 \hat{\rho} \tag{D-13}$$

for,

$$C = \left(\gamma \frac{p_0}{\rho_0}\right)^{\frac{1}{2}} \quad .$$

If the fluid is a liquid, the relationship between $\hat{p}$ and $\hat{\rho}$ given in equation D-13 remains valid but the constant C^2 no longer relates to equilibrium values in exactly the same way. Differentiating twice with respect to time, equation D-13 becomes:

$$\frac{\partial^2 \hat{\rho}}{\partial t^2} = \frac{1}{C^2} \frac{\partial^2 \hat{p}}{\partial t^2} \quad .$$

Eliminating with the aid of equation D-6, the density $\hat{\rho}$

$$\boxed{\frac{1}{C^2} \frac{\partial^2 \hat{p}}{\partial t^2} - \nabla^2 \hat{p} = 0} \tag{D-14}$$

which is the desired wave equation. The constant C, that might in principle depend on $\vec{r}$, represent the wave velocity.

2. The Rytov Method of Approximation

To describe the propagation of an acoustic wave through a turbulent ocean, we must solve the wave equation

$$\nabla^2 p - \frac{N^2}{C_0^2} \frac{\partial^2 p}{\partial t^2} = 0$$

for an index of refraction $N(\vec{r})$ that is a random function of $\vec{r}$. Notice, that here we have regarded the distribution of inhomogeneities as static, neglecting their change as a result of heat conduction, diffusion, convection and drift. We can neglect this change if the propagation time is small compared to the characteristic time scale of change in the inhomogeneities. Correspondingly, if the source is monocromatic, the time dependence of $p(\vec{r},t)$ is of the typical form $e^{i\omega t}$ and the wave equation reduces to the well known Helmholtz equation,

$$\nabla^2 p + k^2 N^2(\vec{r}) p = 0 \tag{D-15}$$

where $k \equiv \frac{\omega}{C_0}$ represents the wave number of the source,

$$k = \frac{2\pi}{\lambda} \quad .$$

We set

$$N(\vec{r}) = 1 + n(\vec{r})$$

with

$$|n(\vec{r})| << 1$$

and initially attempt to solve the Helmholtz equation by the method of small perturbations. The idea behind this method is that due to the smallness of $n(\vec{r})$ the solution $p(\vec{r})$ differs little from the non-turbulent solution and hence, that $p(\vec{r})$ can be looked for in the form of a power series expansion in some small parameter of order n . One can think of the mth term in the expansion as representing the non-turbulent wave scattered m times by the turbulent inhomogeneities of the medium, and hence of the wave actually present at the observation point as the sum of such multiply scattered waves. Indeed, consider the non-turbulent wave $p_0(\vec{r})$ satisfying the homogeneous Helmholtz equation

$$\nabla^2 p_0(\vec{r}) + k^2 p_0(\vec{r}) = 0$$

and allow the medium to develop a turbulent inhomogeneity n in the index of refraction at some arbitrary point $\vec{r}_0$. In the presence of $p_0(\vec{r})$ the inhomogeneity at $\vec{r}_0$ becomes the source of a secondary wave $p_1(\vec{r})$ with source strength $-2k^2 n(\vec{r})p_0(\vec{r})$. The secondary wave coexists with $p_0(\vec{r})$ and satisfies the nonhomogeneous Helmholtz equation,

$$\nabla^2 p_1(\vec{r}) + k^2 p_1(\vec{r}) = -\ 2k^2 n(r) p_0(\vec{r}) \quad .$$

Consequently,

$$p_1(\vec{r}) = \int d\vec{r}_0\ G(\vec{r}|\vec{r}_0) \left[-2k^2 n(\vec{r}_0) p_0(\vec{r}_0)\right]$$

where the Green's function $G(\vec{r}|\vec{r}_0)$ satisfies

$$\nabla^2 G(\vec{r}|\vec{r}_0) + k^2 G(\vec{r}|\vec{r}_0) = -\ \delta(\vec{r}-\vec{r}_0)$$

and represents the contribution to the wave at $\vec{r}$ due to a point source of unit strength at $\vec{r}_0$. The actual field measured at $\vec{r}$ is therefore given by

$$p(\vec{r}) = p_0(\vec{r}) + \int d\vec{r}_0\ G(\vec{r}|\vec{r}_0) \left[-2k^2 n(\vec{r}_0) p_0(\vec{r}_0)\right] \quad .$$

Let now another inhomogeneity develop at $\vec{r}_1$. In the presence of $p_1(\vec{r})$, it will become the source of a secondary wave of strength $-2k^2 n(\vec{r}) p_1(\vec{r})$ and hence,

$$p_2(\vec{r}) = \int d\vec{r}_1\ G(\vec{r}|\vec{r}_1) \left[-2k^2 n(\vec{r}_1) p_1(\vec{r}_1)\right] \quad .$$

By repeating the argument ad infinitum, it is not difficult to show that the wave actually measured at $\vec{r}$ can be written as a multiple scattering expansion:

$$p(\vec{r}) = p_0(\vec{r}) + \sum_{m=1}^{\infty} p_m(\vec{r})$$

where

$$p_m(\vec{r}) = \int d\vec{r}_{m-1}\ G(\vec{r}|\vec{r}_{m-1}) \left[-2k^2 n(\vec{r}_{m-1}) p_{m-1}(\vec{r}_{m-1})\right] \quad .$$

The iterative nature of this equation indicates that the m$\underline{th}$ term in the expansion is of order n^m. Therefore, for sufficiently small n one might hope that higher terms in the multiple scattering expansion can be neglected with impunity. Since, however, one expects the fluctuations away from the nonturbulent value to grow with the range, a straightforward application of the method of small perturbations shall not suffice for all but very short ranges. Rather, [7,8] we write equation D-15 as,

$$\frac{\nabla^2 p}{p} + k^2 N^2 = 0$$

and notice that it is equivalent to,

$$\nabla^2 \log p + (\nabla \log p)^2 + k^2 N^2 = 0 \quad . \tag{D-16}$$

We set

$$\psi \equiv \log p$$

so that

$$\psi = \log A + iS$$

where A represents the amplitude of p, S the phase thereof, and have,

$$\nabla^2 \psi + (\nabla\psi)^2 + k^2 N^2 = 0 \quad . \tag{D-17}$$

We are now ready to apply the method of small perturbations to this modified differential equation by taking, to first order,

$$\psi = \psi_0 + \psi_1 \tag{D-18}$$

where the zeroth approximation

$$\psi_0 = \log A_0 + iS_0$$

satisfies:

$$\nabla^2 \psi_0 + (\nabla\psi_0)^2 + k^2 = 0 \tag{D-19}$$

and where ψ_1 is the once scattered wave. Then, using equations D-17, 18, 19

$$\nabla^2 \psi_1 + \nabla\psi_1(2\nabla\psi_0 + \nabla\psi_1) + 2k^2n + k^2n^2 = 0 \tag{D-20}$$

where

$$\psi_1 = \log \frac{A}{A_0} + i(S-S_0) \quad .$$

In equation D-20, we omit k^2n^2 which is of second order of smallness and further assume that

$$|\nabla\psi_1| << |\nabla\psi_0| \quad . \tag{D-21}$$

Hence, the equation to be solved becomes:

$$\nabla^2 \psi_1 + 2\nabla\psi_0 \cdot \nabla\psi_1 + 2k^2n = 0 \quad . \tag{D-22}$$

Since,

$$|\nabla\psi_0| \sim k$$

condition D-21 describing the limits of validity of our approximation now reads:

$$\lambda|\nabla\psi_1| << 2\pi \tag{D-23}$$

and expresses the smallness of the change of ψ_1 over distances of the order of a wave length. This inequality implies the smallness of the relative amplitude change over a wave length

$$\lambda\left|\nabla \log \frac{A}{A_0}\right| << 2\pi \tag{D-24}$$

and the smallness of the change of the phase fluctuation over λ ,

$$\lambda|\nabla(S-S_0)| << 2\pi \quad . \tag{D-25}$$

The amplitude condition D-24 is always fulfilled in a weakly inhomogeneous medium, $|n| << 1$. The phase condition D-25 implies that the angle of inclination of the ray to the initial direction is small. In fact, if

$$S_0 = kX$$

we have,

$$\frac{\partial S}{\partial X} = k + \frac{\partial S_1}{\partial X} \quad ; \quad \frac{\partial S}{\partial y} = \frac{\partial S_1}{\partial y} \quad ; \quad \frac{\partial S}{\partial z} = \frac{\partial S_1}{\partial z}$$

for

$$S_1 \equiv S - S_0$$

and hence, using equation D-25,

$$\left|\frac{\partial S}{\partial X}\right| \sim k \quad ; \quad \left|\frac{\partial S}{\partial y}\right| << k \quad ; \quad \left|\frac{\partial S}{\partial Z}\right| << k \quad .$$

Since large scale inhomogeneities, $\lambda << \Lambda_0$, produce sharply directed forward scattering, the phase condition D-25 can be met if we assume:

$$\frac{\lambda}{\Lambda_0} << 1 \quad .$$

3. The Method of Spectral Expansions

We now consider the problem of fluctuations of a monocromatic wave, confining ourselves to the case where the wave length λ is small compared to the inner scale of turbulence Λ_0 . In this case, the angle of scattering of the wave by refractive index inhomogeneities is of order no greater than λ / Λ_0 and is thus small. Therefore, the value of $\psi_1(\vec{r})$ can only be appreciably affected by the inhomogeneities included in a cone with vertex at the observation point, with axis directed towards the wave source and with angular aperture λ / Λ_0 . Hence inhomogeneities lying on the opposite side of the source from the observation point never contribute to scattering and we shall therefore take $n(\vec{r}) = 0$ in that region. To simplify the calculation we shall furthermore assume that the wave is plane. We then locate the origin of coordinates on the boundary of the region occupied by the refractive index inhomogeneities and direct the x-axis along the direction of propagation of the incident wave.

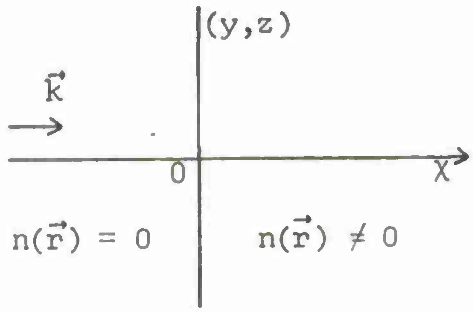

FIG. 2. THE DISTRIBUTION OF REFRACTIVE INDEX INHOMOGENEITIES

Then,

$$\nabla \psi_0 = (ik, 0, 0)$$

and the differential equation D-22 reduces to,

$$\nabla^2 \psi_1 + 2ik \frac{\partial \psi_1}{\partial X} + 2k^2 n = 0 \quad .$$

If $\lambda << \Lambda_0$, one can show that it shall suffice to solve the simpler equation,

$$\nabla^2_{\perp} \psi_1 + 2i \frac{\partial \psi_1}{\partial X} + 2k^2 n = 0 \qquad \text{(D-26)}$$

where

$$\nabla^2_{\perp} \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad .$$

The method of spectral expansion consists in taking

$$n(\vec{r}) = \int e^{i\vec{\varkappa}\cdot\vec{r}} \, dZ(\vec{\varkappa}) \quad . \qquad \text{(D-27)}$$

and searching for the solution in the form,

$$\psi_1(\vec{r}) = \int e^{i\vec{\varkappa}\cdot\vec{r}}\, d\varphi\,(\vec{\varkappa}) \quad . \tag{D-28}$$

Using the differential equation D-26,

$$\int e^{i\vec{\varkappa}\cdot\vec{r}} \left(-\varkappa^2_{\perp} - 2k\varkappa_{\parallel}\right)\, d\varphi(\vec{\varkappa}) + 2k^2 \int e^{i\vec{\varkappa}\cdot\vec{r}}\, dZ(\vec{\varkappa}) = 0$$

and therefore,

$$d\varphi(\vec{\varkappa}) = \frac{k}{\vec{\varkappa}^2_{\perp}/2k + \varkappa_{\parallel} - i\epsilon}\, dZ\,(\vec{\varkappa}) \quad . \tag{D-29}$$

Notice the small imaginary part that has been added to the denominator to render the expression meaningful at $\varkappa_{\parallel} = -\varkappa^2_{\perp}/2k$. Before proceeding any further, we must first explicitly incorporate the boundary condition

$$n(\vec{r}) = 0 \quad , \quad X < 0$$

into the Fourier-Stieltjes transform D-27 of the refractive index field. We therefore take,

$$n(\vec{r}) = \theta(X)\eta(\vec{r}) \qquad .$$

with the step function $\theta(X)$ defined by

$$\theta(X) \equiv \begin{cases} 1 & , \quad X > 0 \\ 0 & , \quad X < 0 \end{cases}$$

and invert the Fourier-Stieltjes transformation D-27 to have,

$$dZ(\vec{\varkappa}) = -\frac{1}{2\pi}\, d\varkappa_{\parallel} \int d\nu(\omega, \vec{\varkappa}_{\perp}) \int_0^{\infty} dX e^{i(\omega-\varkappa_{\parallel})X} \tag{D-30}$$

where $d\nu(\vec{\kappa})$ is the Fourier Stieltjes transform of $\eta(\vec{r})$.

Using equation D-30 we can now formally express the solution $\psi_1(\vec{r})$ in terms of the known Fourier-Stieltjes transform of $\eta(\vec{r})$. Thus, we can express the transform of the random field of logorithmic amplitude fluctuations,

$$\mathrm{Re}\,\psi_1(\vec{r}) = \int e^{i\vec{\kappa}\cdot\vec{r}}\, da(\vec{\kappa})$$

in terms of $d\nu(\vec{\kappa})$ using the obvious relation

$$da(\vec{\kappa}) = \frac{1}{2}\left[d\varphi(\vec{\kappa}) + d\varphi^*(-\vec{\kappa})\right]$$

and obtain

$$da(\vec{\kappa}) = -\frac{k}{4\pi}\, d\kappa_{\parallel} \int_0^{\infty} d\nu(\omega,\vec{\kappa}_{\perp}) \left[\frac{1}{\kappa_{\perp}^2/2k + \kappa_{\parallel} - i\epsilon} \int_0^{\infty} dX\; e^{i(\omega-\kappa_{\parallel})X}\right.$$

$$\left. + \frac{1}{\kappa_{\perp}^2/2k - \kappa_{\parallel} + i\epsilon} \int_0^{\infty} dX e^{i(\omega-\kappa_{\parallel})X}\right] \tag{D-31}$$

We can also express the Fourier-Stieltjes transform of the random field of phase fluctuations

$$\mathrm{Im}\,\psi_1(\vec{r}) = \int e^{i\vec{\kappa}\cdot\vec{r}}\, d\sigma(\vec{\kappa})$$

in terms of $d\nu(\vec{\kappa})$ using,

$$d\sigma(\vec{\kappa}) = \frac{1}{2i}\left[d\varphi(\vec{\kappa}) - d\varphi^*(-\vec{\kappa})\right] .$$

We get,

$$d\sigma(\vec{\varkappa}) = -\frac{k}{4\pi i}\, d\varkappa_{\parallel} \int d\nu(\omega,\varkappa_{\perp}) \Big[\frac{1}{\varkappa_{\perp}^2/2k + \varkappa_{\parallel} - i\epsilon} \int_0^{\infty} dX e^{i(\omega-\varkappa_{\parallel})X}$$

$$- \frac{1}{\varkappa_{\perp}^2/2k - \varkappa_{\parallel} + i\epsilon} \int_0^{\infty} dX e^{i(\omega-\varkappa_{\parallel})X} \Big] \qquad \text{(D-32)}$$

Correspondingly, we can now calculate the correlation function of logorithmic amplitude fluctuations in terms of the known correlation function of the refractive index field, or the Fourier transform thereof. Indeed,

$$B_A(\vec{\rho}) \equiv \langle \operatorname{Re}\psi_1(\vec{r}+\vec{\rho}) \operatorname{Re}^*\psi_1(\vec{r}) \rangle =$$

$$= \int e^{i\vec{\varkappa}(\vec{r}+\vec{\rho})}\, e^{-i\vec{\varkappa}'\vec{r}} \langle da(\vec{\varkappa}) da^*(\vec{\varkappa}') \rangle$$

and recognizing that

$$\langle da(\vec{\varkappa}) da^*(\vec{\varkappa}') \rangle = \delta(\vec{\varkappa}_{\perp} - \vec{\varkappa}'_{\perp})\; \Phi_A(\varkappa_{\parallel}, \varkappa'_{\parallel}; \vec{\varkappa}_{\perp}) d\varkappa_{\parallel} d\varkappa'_{\parallel} d\vec{\varkappa}_{\perp} d\vec{\varkappa}'_{\perp}$$

we have,

$$B_A(\vec{\rho}) = \int_{-\infty}^{\infty} d\varkappa_{\parallel} d\varkappa'_{\parallel} d\vec{\varkappa}_{\perp}\, e^{i\vec{\varkappa}_{\perp}\vec{\rho}_{\perp}}\, e^{i\varkappa_{\parallel}(r_{\parallel}+\rho_{\parallel})}\, e^{-i\varkappa'_{\parallel} r_{\parallel}}\, \Phi_A(\varkappa_{\parallel}, \varkappa'_{\parallel}; \vec{\varkappa}_{\perp}) \quad . \qquad \text{(D-33)}$$

The spectal density function $\Phi_A(\varkappa_{\parallel}, \varkappa'_{\parallel}; \varkappa_{\perp})$ is easily connected to the spectral density function of the refractive index field via its definition and equation D-31,

$$\Phi_A(\varkappa_{\parallel}, \varkappa'_{\parallel}; \vec{\varkappa}_{\perp}) = \frac{k^2}{(4\pi)^2} \int_{-\infty}^{\infty} d\omega\; \Phi_n(\omega, \vec{\varkappa}_{\perp})\, *$$

$$\Big[\frac{1}{(\varkappa_{\perp}^2/2k + \varkappa_{\parallel} - i\epsilon)(\varkappa_{\perp}^2/2k + \varkappa'_{\parallel} + i\epsilon)} \int_0^{\infty} dX dX' e^{i(\omega-\varkappa_{\parallel})X}\, e^{-i(\omega-\varkappa'_{\parallel})X'}$$

$$+ \frac{1}{(\kappa_\perp^2/2k+\kappa_{/\!/}-i\epsilon)(\kappa_\perp^2/2k-\kappa'_{/\!/}-i\epsilon)} \int_0^\infty dXdX' e^{i(\omega-\kappa_{/\!/})X} e^{-i(\omega-\kappa'_{/\!/})X'}$$

$$+ \frac{1}{(\kappa_\perp^2/2k-\kappa_{/\!/}+i\epsilon)(\kappa_\perp^2/2k+\kappa'_{/\!/}+i\epsilon)} \int_0^\infty dXdX' e^{i(\omega-\kappa_{/\!/})X} e^{-i(\omega-\kappa'_{/\!/})X'} \quad \text{(D-34)}$$

$$+ \frac{1}{(\kappa_\perp^2/2k-\kappa_{/\!/}+i\epsilon)(\kappa_\perp^2/2k-\kappa'_{/\!/}-i\epsilon)} \int_0^\infty dXdX' e^{i(\omega-\kappa_{/\!/})X} e^{-i(\omega-\kappa'_{/\!/})X'} \Bigg]$$

where,

$$\langle d\nu(\omega,\vec{\kappa}_\perp) d\nu^*(\omega',\kappa'_\perp)\rangle = \delta(\omega-\omega')\delta(\vec{\kappa}_\perp-\vec{\kappa}'_\perp)\, \Phi_n(\omega,\kappa_\perp) d\omega d\omega' d\vec{\kappa}_\perp d\vec{\kappa}'_\perp \quad .$$

We are now in a position to evaluate the $\kappa_{/\!/}, \kappa'_{/\!/}$ integrals in equation D-33. To do so, define:

$$F_A(\alpha,\beta;\vec{\kappa}_\perp) \equiv \int_{-\infty}^{\infty} d\kappa_{/\!/} d\kappa'_{/\!/}\, e^{i\alpha\kappa_{/\!/}} e^{-i\beta\kappa'_{/\!/}}\, \Phi_A(\kappa_{/\!/},\kappa'_{/\!/};\vec{\kappa}_\perp) \quad .$$

Replacing Φ_A from equation D-34, interchanging the (ω, X, X') integrals with the $(\kappa_{/\!/}, \kappa')$ integration, and employing repeatedly the spectral representation of the step function,

$$\theta(t) = \lim_{\epsilon\to 0} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega+i\epsilon}$$

we obtain:

$$F_A(\alpha, \beta; \vec{\varkappa}_\perp) = \frac{1}{2} k^2 \int_0^\alpha dX \int_0^\beta dX' \; F_n(X-X'; \vec{\varkappa}_\perp) \, * \tag{D-35}$$

$$\left[\cos\left(\frac{\varkappa_\perp^2}{2k}(\beta-\alpha+X-X')\right) - \cos\left(\frac{\varkappa_\perp^2}{2k}(\beta+\alpha-X-X')\right)\right]$$

where,

$$F_n(X-X'; \vec{\varkappa}_\perp) \equiv \int_{-\infty}^{\infty} d\omega \, e^{i\omega(X-X')} \, \Phi_n(\omega; \vec{\varkappa}_\perp) \quad .$$

In equation D-35, the upper limit of both integrals can be set equal to

$$\gamma = \text{Min}(\alpha, \beta) \quad .$$

In fact, because of the directional characters of the scattering, the waves scattered by the layer bounded by the planes $X = \text{Min}(\alpha, \beta)$ and $X = \text{Max}(\alpha, \beta)$ are incident on the receiver at $\text{Max}(\alpha, \beta)$ but not on the receiver at $\text{Min}(\alpha, \beta)$. Therefore, these waves can be neglected in the calculation of the correlation function.

Introducing,

$$\xi \equiv X-X' \quad ; \qquad \eta \equiv X+X'$$

$F_A(\alpha, \beta; \vec{\varkappa}_\perp)$ becomes:

$$F_A(\alpha,\beta;\vec{\varkappa}_\perp) = \frac{1}{2}k^2 \int_0^\gamma d\xi\, F_n(\xi;\vec{\varkappa}_\perp)\,(\gamma-\xi)\, *$$

$$\left[\cos\left(\frac{\varkappa_\perp^2}{2k}(\beta-\alpha-\xi)\right) + \cos\left(\frac{\varkappa_\perp^2}{2k}(\beta-\alpha+\xi)\right)\right]$$

(D-36)

$$-\frac{1}{2}k^2 \int_0^\gamma d\xi\, F_n(\xi;\vec{\varkappa}_\perp)\,\frac{2k}{\varkappa_\perp^2}\, *$$

$$\left[\sin\left(\frac{\varkappa_\perp^2}{2k}(\beta+\alpha-\xi)\right) - \sin\left(\frac{\varkappa_\perp^2}{2k}(\beta+\alpha-2\gamma+\xi)\right)\right] .$$

In general, the function $F_n(\xi;\vec{\varkappa}_\perp)$ falls off very rapidly to zero for $\varkappa_\perp\xi \geq 1$. Therefore, the important contribution to the values of the integrals occurs for $\xi \leq \frac{1}{\varkappa_\perp}$. In the region $\xi \leq \frac{1}{\varkappa_\perp}$, we have $\frac{\varkappa_\perp^2 \xi}{k} \leq \frac{\varkappa_\perp}{k}$. We assumed above that the wavelength λ is much less than the inner scale of turbulence Λ_0. But $\Lambda_0 \sim \frac{1}{\varkappa_{\perp m}}$, where $\varkappa_{\perp m}$ is the largest wave number for which $F_n(\xi;\vec{\varkappa}_\perp)$ still differs from zero. Therefore we have:

$$\frac{\varkappa_\perp}{k} < \frac{\varkappa_{\perp m}}{k} << 1 .$$

Thus, $\frac{\varkappa_\perp^2 \xi}{2k} << 1$ in the important region of integration on ξ.

Furthermore, we shall be interested in the correlation function of $\mathrm{Re}\,\psi(\vec{r})$ only for values of the argument which are small compared to the range γ. This means that in equation D-36 we consider only values of $\varkappa_\perp$ which satisfy the condition $\frac{1}{\varkappa_\perp} < \gamma$. Since $\xi \leq \frac{1}{\varkappa_\perp}$ in the important region of integration, within this region we have $\xi << \gamma$.

Taking all these simplifications into account, we obtain:

$$F_A(\alpha, \beta; \vec{\kappa}_\perp) \approx k^2 \Big[\gamma \cos\left(\frac{\kappa_\perp^2}{2k}(\beta-\alpha)\right) + \frac{k}{\kappa_\perp^2}\sin\left(\frac{\kappa_\perp^2}{2k}(\beta+\alpha-2\gamma)\right)$$

$$- \frac{k}{\kappa_\perp^2}\sin\left(\frac{\kappa_\perp^2}{2k}(\beta+\alpha)\right)\Big] \int_0^\gamma d\xi\, F_n(\xi; \vec{\kappa}_\perp)$$

Recognizing that,

$$\int_0^\gamma d\xi\, F_n(\xi;\vec{\kappa}_\perp) \approx \int_0^\infty d\xi\, F_n(\xi;\vec{\kappa}_\perp) = \pi\, \Phi_n(0;\vec{\kappa}_\perp)$$

we finally have:

$$\boxed{F_A(\alpha, \beta; \vec{\kappa}_\perp) = k^2 \pi \gamma \Phi_n(0;\vec{\kappa}_\perp) \Big[\cos\left(\frac{\kappa_\perp^2}{2k}(\beta-\alpha)\right) + \frac{k}{\kappa_\perp^2 \gamma}\sin\left(\frac{\kappa_\perp^2}{2k}(\beta+\alpha-2\gamma)\right) - \frac{k}{\kappa_\perp^2 \gamma}\sin\left(\frac{\kappa_\perp^2}{2k}(\beta+\alpha)\right)\Big]} \tag{D-37}$$

Now, in accordance with equation D-33, the correlation function of the logarithmic amplitude field is given by,

$$B_A(\vec{\rho}) \equiv \int d\vec{\kappa}_\perp\, e^{i\vec{\kappa}_\perp \vec{\rho}_\perp} F_A(r_{/\!/}+\rho_{/\!/}, r_{/\!/}; \vec{\kappa}_\perp) \quad .$$

Choosing one of the two $\vec{\kappa}_\perp$ axis along $\vec{\rho}_\perp$, we can perform the angular integration to have:

$$B_A(\vec{\rho}) \equiv 2\pi \int_0^\infty \varkappa_\perp d\varkappa_\perp J_0(\varkappa_\perp \rho_\perp)\, F_A(r_{/\!/}+\rho_{/\!/}, r_{/\!/}; \varkappa_\perp) \quad .$$

We are therefore interested in,

$$\alpha \equiv r_{/\!/} + \rho_{/\!/} \qquad ; \qquad \beta = r_{/\!/}$$

and with the notation of figure 3 we thus have

$$\beta - \alpha = \rho \sin\theta \quad ; \qquad \beta + \alpha = 2L$$

$$\rho_\perp = \rho \cos\theta$$

$$\gamma = L - \frac{1}{2}\rho \sin\theta$$

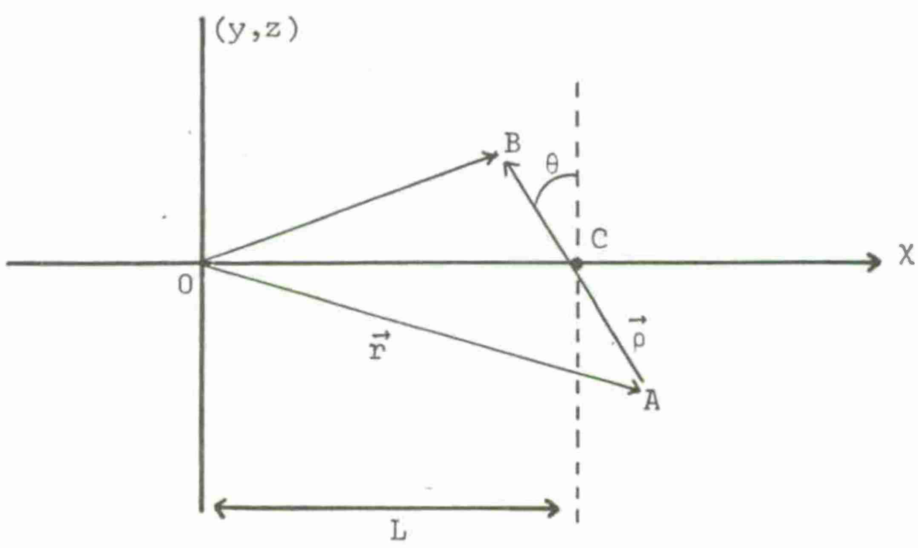

FIG. 3. THE GEOMETRY OF THE CORRELATION FUNCTION MEASUREMENT

Then,

$$B_A(\vec{\rho}) = 2\pi^2 k^2 \,*$$

$$\Bigg[(L - \tfrac{1}{2}\rho \sin\theta)\int_0^\infty \varkappa_\perp d\varkappa_\perp J_0(\varkappa_\perp \rho\cos\theta)\cos\left(\frac{\varkappa_\perp^2}{2k}\rho\sin\theta\right)\Phi_n(0;\varkappa_\perp)$$

$$+ k\int_0^\infty \varkappa_\perp d\varkappa_\perp J_0(\varkappa\ \rho\cos\theta)\,\frac{\sin\left(\frac{\varkappa_\perp^2}{2k}\rho\sin\theta\right)}{\varkappa_\perp^2}\,\Phi_n(0;\varkappa_\perp) \qquad \text{(D-38)}$$

$$- k\int_0^\infty \varkappa_\perp d\varkappa_\perp J_0(\varkappa_\perp \rho\cos\theta)\,\frac{\sin\left(\frac{\varkappa_\perp^2}{k}L\right)}{\varkappa_\perp^2}\,\Phi_n(0;\varkappa_\perp)\Bigg]$$

Notice that $B_A(\vec{\rho})$ does depend on the range L, thus indicating that the field $\mathrm{Re}\ \psi_1(\vec{r})$ is not homogeneous. This circumstance reflects the nonhomogeneous choice of "boundaries" implicit in the assumption of forward scattering and is not characteristic of the field itself.

4. PHENOMENOLOGICAL MODEL

We are now ready to choose some reasonable model for the Fourier transform $\Phi_n(\vec{\varkappa})$ of the correlation function characterizing the refractive index inhomogeneities. A natural candidate for the structure function of a turbulent field has already been suggested in chapter B. Here, however, with the field assumed homogeneous and isotropic rather than locally so, the structure function can be eliminated in favor of the correlation function

$$B(\rho) = \frac{1}{2}\left[D(\infty) - D(\rho)\right] .$$

Using the Karman proposal, we therefore have,

$$B_n(\rho) = \frac{2^{2/3}}{\Gamma(1/3)} < n^2 > \left(\frac{\rho}{L_0}\right)^{1/3} K_{1/3}\left(\frac{\rho}{L_0}\right)$$

and correspondingly,

$$\Phi_n(\vec{\kappa}) = \frac{\Gamma(11/6)}{\pi^{3/2}\Gamma(1/3)} L_0^3 < n^2 > \frac{1}{(1+\kappa^2 L_0^2)^{11/6}} \quad .$$

In what follows, we shall find it convenient to write

$$\Phi_n(\vec{\kappa}) = \lim_{\gamma\to 0^+} \frac{\Gamma(11/6)}{\pi^{3/2}\Gamma(1/3)} L_0^3 < n^2 > \frac{e^{-\gamma\kappa^2}}{(1+\kappa^2 L_0^2)^{11/6}} \tag{D-39}$$

and postpone the taking of the limit to the end of our calculation. With this choice for $\Phi_n(\vec{\kappa})$, the correlation function for the logarithmic amplitude becomes:

$$B_A(\vec{\rho}) = 2\pi^{\frac{1}{2}} < n^2 > k^3 L_0^3 \frac{\Gamma(11/6)}{\Gamma(1/3)} \lim_{\gamma\to 0^+} \left[\frac{L}{k} A_1 - A_2 - \frac{1}{2k} \rho \sin\theta \, A_1 + A_3\right] \tag{D-40}$$

where

$$A_1 \equiv \int_0^\infty \kappa_\perp d\kappa_\perp J_0(\kappa_\perp \rho \cos\theta) \cos\left(\frac{\kappa_\perp^2}{2k} \rho \sin\theta\right) \frac{e^{-\gamma\kappa_\perp^2}}{(1+\kappa_\perp^2 L_0^2)^{11/6}}$$

$$A_2 \equiv \int_0^\infty \kappa_\perp d\kappa_\perp J_0(\kappa_\perp \rho \cos\theta) \frac{\sin\left(\frac{\kappa_\perp^2}{k} L\right)}{\kappa_\perp^2} \frac{e^{-\gamma\kappa_\perp^2}}{(1+\kappa_\perp^2 L_0^2)^{11/6}}$$

$$A_3 \equiv \int_0^\infty \kappa_\perp d\kappa_\perp J_0(\kappa_\perp \rho \cos\theta) \frac{\sin\left(\frac{\kappa_\perp^2}{2k} \rho \sin\theta\right)}{\kappa_\perp^2} \frac{e^{-\gamma\kappa_\perp^2}}{(1+\kappa_\perp^2 L_0^2)^{11/6}} \quad .$$

We begin the evaluation of A_1 recognizing that

$$A_1 = \text{Re } I(z)$$

for

$$I(z) \equiv \int_0^\infty \kappa_\perp d\kappa_\perp e^{-z\kappa_\perp^2} \frac{J_0(\beta\kappa_\perp)}{(1+\kappa_\perp^2 L_0^2)^{11/6}}$$

and where,

$$z \equiv \gamma - i\alpha \; ; \qquad \text{Re}z > 0$$

$$\alpha \equiv \frac{1}{2}k \; \rho \sin\theta$$

$$\beta \equiv \rho \cos\theta$$

Changing variables, we can write,

$$I(z) = \frac{1}{2L_0^2} \int_0^\infty dx \; e^{-zx/L_0^2} \frac{J_0\left(\frac{\beta}{L_0} x^{\frac{1}{2}}\right)}{(1+x)^{11/6}}$$

and using the power series expansion

$$J_0\left(\frac{\beta}{L_0} x^{\frac{1}{2}}\right) = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \left(\frac{\beta^2}{4L_0^2}\right)^n x^n$$

we have:

$$I(z) = \frac{1}{2L_0^2} \sum_{n=0}^{\infty} \frac{(-1)^2}{(n!)^2} \left(\frac{\beta^2}{4L_0^2}\right)^n \int_0^{\infty} dx\; e^{-zx/L_0^2} (1+x)^{-11/6}\, x^n$$

$$\text{(D-41)}$$

$$I(z) = \frac{1}{2L_0^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\beta^2}{4L_0^2}\right)^n \Psi\left(n+1 \,\middle|\, n+\frac{1}{6} \,\middle|\, \frac{z}{L_0^2}\right)$$

where $\Psi(a\,|\,c\,|\,x)$ represents the confluent hypergeometric function of Tricomi[9] satisfying the integral representation,

$$\Psi(a\,|\,c\,|\,x) = \frac{1}{\Gamma(a)} \int_0^{\infty} dt\; e^{-xt}\, t^{a-1}\, (1+t)^{c-a-1}$$

$$\mathrm{Re}\,a > 0 \quad , \qquad \mathrm{Re}\,x > 0 \quad .$$

If $\left|\frac{z}{L_0^2}\right| << 1$, we can use the behavior near the origin for the Tricomi function,

$$\Psi\left(1\,\middle|\,\frac{1}{6}\,\middle|\,\frac{2}{L_0^2}\right) \to \frac{\Gamma(5/6)}{\Gamma(11/6)} + \Gamma(-5/6)\left(\frac{z}{L_0^2}\right)^{5/6}$$

$$\psi(n+1\,|\,n+1/6\,|\,z/L_0^2) \to \frac{\Gamma(n-5/6)}{\Gamma(n+1)} \left(\frac{z}{L_0^2}\right)^{5/6-n}$$

$$n \geq 1$$

to obtain,

$$I(z) \to \frac{1}{2L_0^2} \frac{\Gamma(5/6)}{\Gamma(11/6)} \frac{1}{2L_0^2} \Gamma(-5/6) \left(\frac{z}{L_0^2}\right)^{5/6} +$$

$$+ \frac{1}{2L_0^2} \left(\frac{z}{L_0^2}\right)^{5/6} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\beta^2}{4z}\right)^n \Gamma(n-5/6) \quad .$$

Employing the standard integral representation of the Euler gamma function

$$\Gamma(n-5/6) = \int_0^\infty dte^{-t}\, t^{n-11/6} \quad , \qquad (n-5/6) > 0$$

$$I(z) \to \frac{1}{2L_0^2} \frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{1}{2L_0^2} \Gamma(-5/6) \left(\frac{z}{L_0^2}\right)^{5/6}$$

$$+ \frac{1}{2L_0^2} \left(\frac{z}{L_0^2}\right)^{5/6} \int_0^\infty dte^{-t}\, t^{-11/6} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\beta^2 t}{4z}\right)^n$$

and recognizing that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\beta^2 t}{4z}\right)^n = J_0 \left[2\left(\frac{\beta^2 t}{4z}\right)^{\frac{1}{2}}\right] - 1$$

it follows

$$I(z) \to \frac{1}{2L_0^2}\frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{1}{2L_0^2}\,\Gamma(-5/6)\left(\frac{z}{L_0^2}\right)^{5/6} +$$

$$+ \frac{1}{2L_0^2}\left(\frac{z}{L_0^2}\right)^{5/6} \int_0^\infty dte^{-t}\ t^{-11/6}\left(J_0\left[2\left(\frac{\beta^2 t}{4z}\right)^{\frac{1}{2}}\right] -1\right) \quad .$$

Consider now,

$$D\left(\frac{\beta^2}{4z}\right) \equiv \int_0^\infty dte^{-t}\ t^{-11/6}\left(J_0\left[2\left(\frac{\beta^2 t}{4z}\right)^{\frac{1}{2}}\right] -1\right)$$

and use,

$$J_0\left[2\left(\frac{\beta^2 t}{4z}\right)^{\frac{1}{2}}\right] -1 = -t^{\frac{1}{2}} \int_0^{2\left(\frac{\beta^2}{4z}\right)^{\frac{1}{2}}} d\xi\ J_1(\xi t^{\frac{1}{2}})$$

to have:

$$D\left(\frac{\beta^2}{4z}\right) = - \int_0^{2\left(\frac{\beta^2}{4z}\right)^{\frac{1}{2}}} d\xi \int_0^\infty dte^{-t}\ t^{-8/6}\ J_1\left[2\left(\frac{\xi^2 t}{4}\right)^{\frac{1}{2}}\right] \quad .$$

Notice, however, that

$$\int_0^\infty dte^{-t}\ t^{-8/6}\ J_1\left[2\left(\frac{\xi^2 t}{4}\right)^{\frac{1}{2}}\right] = \frac{\Gamma(1/6)}{\Gamma(2)}\ e^{-\xi^2/4}\ \frac{\xi}{2}\ \Phi\left(11/6\,|\,2\,|\,\xi^2/4\right)$$

where $\Phi(a|c|x)$ represent the confluent hypergeometric function of Kummer.[9] Hence,

$$D\left(\frac{\beta^2}{4z}\right) = \frac{\Gamma(1/6)}{\Gamma(2)} \int_0^{\beta^2/4z} dx\ e^{-x}\ \Phi\left(11/6|2|x\right)$$

and upon using

$$\frac{d}{dx}\left[e^{-x}\ \Phi\ (11/6|1|x)\right] = -\frac{\Gamma(1/6)\Gamma(1)}{\Gamma(-5/6)\Gamma(2)}\ e^{-x}\ \Phi\ (11/6|2|x)$$

we find,

$$D\left(\frac{\beta^2}{4z}\right) = \Gamma(-5/6)\left[e^{-\beta^2/4z}\ \Phi\ (11/6|1|\beta^2/4z) -1\right] .$$

Thus:

$$I(z) \to \frac{1}{2L_0^2}\frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{1}{2L_0^2}\Gamma(-5/6)\left(\frac{z}{L_0^2}\right)^{5/6} e^{-\beta^2/4z}\ \Phi\ (11/6|1|\beta^2/4z)$$

or,

$$\boxed{I(z) \to \frac{1}{2L_0^2}\frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{1}{2L_0^2}\Gamma(-5/6)\left(\frac{z}{L_0^2}\right)^{5/6} \Phi\ (-5/6|1|-\beta^2/4z)} \qquad \text{(D-42)}$$

$$\left|\frac{z}{L_0^2}\right| \ll 1$$

where use has been made of the Kummer transformation,

$$\Phi(a|c|x) = e^{x}\ \Phi\ (c-a|c|-x) .$$

The real part of equation D-42 is then the value of the A_1 integral we have been trying to evaluate.

Consider next,

$$G(\gamma, \alpha) \equiv \int_0^\infty \varkappa_\perp d\varkappa_\perp e^{-\gamma \varkappa_\perp^2} \frac{\sin(\alpha \varkappa_\perp^2)}{\varkappa_\perp^2} \frac{J_0(\beta \varkappa_\perp)}{(1+\varkappa_\perp^2 L_0^2)^{11/6}} .$$

Notice that $G(\gamma, \alpha)$ has been so chosen as to provide

$$A_2 = G(\gamma, L/K) \quad ; \qquad A_3 = G(\gamma, \rho/2k \sin\theta)$$

thus reducing both A_2 and A_3 to an evaluation of $G(\gamma, \alpha)$. With

$$\frac{\sin(\alpha \varkappa_\perp^2)}{\varkappa_\perp^2} = \alpha \int_0^1 d\lambda \cos(\lambda \alpha \varkappa_\perp^2)$$

we have:

$$G(\gamma, \alpha) = \frac{\alpha}{2} \int_0^1 d\lambda \int_0^\infty \varkappa_\perp d\varkappa_\perp e^{-(\gamma - i\alpha\lambda)\varkappa_\perp^2} \frac{J_0(\beta \varkappa_\perp)}{(1+\varkappa_\perp^2 L_0^2)^{11/6}}$$

$$+ \frac{\alpha}{2} \int_0^1 d\lambda \int_0^\infty \varkappa_\perp d\varkappa_\perp e^{-(\gamma + i\alpha\lambda)\varkappa_\perp^2} \frac{J_0(\beta \varkappa_\perp)}{(1+\varkappa_\perp^2 L_0^2)^{11/6}}$$

and hence;

$$G(\gamma, \alpha) = \frac{\alpha}{2} \int_0^1 d\lambda \left[I(\gamma - i\alpha\lambda) + I(\gamma + i\alpha\lambda) \right]$$

$$G(\gamma, \alpha) = \frac{1}{2i} \int_{\gamma - i\alpha}^{\gamma + i\alpha} dz\, I(z)$$

$$G(\gamma, \alpha) = \mathrm{Im}\, K(z)$$

where,

$$K(z) \equiv \int dz\, I(z) \quad .$$

Using the expression for I(z) obtained in equation D-42

$$K(z) \rightarrow \frac{1}{2} \frac{\Gamma(5/6)}{\Gamma(11/6)} \left(\frac{z}{L_0^2}\right) + \frac{1}{2L_0^2} \Gamma(-5/6) \left(\frac{\beta^2}{4L_0^2}\right)^{5/6}$$

$$\int dz \left(\frac{4z}{\beta^2}\right)^{5/6} e^{-\beta^2/4z}\, \Phi\,(11/6 | 1 | \beta^2/4z) \quad .$$

Denoting,

$$D(z) \equiv \int dz \left(\frac{4z}{\beta^2}\right)^{5/6} e^{-\beta^2/4z}\, \Phi\,(11/6 | 1 | \beta^2/4z)$$

it follows that,

$$D(z) = - \frac{\beta^2}{4} \int d\left(\frac{\beta^2}{4z}\right) \left(\frac{\beta^2}{4z}\right)^{-17/6} e^{-\beta^2/4z} \, \Phi\,(11/6|1|\beta^2/4z)$$

and using

$$\frac{d}{dx}\left[x^{-11/6} e^{-x} \, \Phi\,(17/6|1|x)\right] = \frac{\Gamma(-5/6)}{\Gamma(-11/6)} \, x^{-17/6} e^{-x} \, \Phi(11/6|1|x)$$

we obtain:

$$D(z) = - \frac{\beta^2}{4} \frac{\Gamma(-11/6)}{\Gamma(-5/6)} \left(\frac{\beta^2}{4z}\right)^{-11/6} e^{-\beta^2/4z} \, \Phi\,(17/6|1|\beta^2/4z) \quad .$$

Hence,

$$\boxed{K(z) \rightarrow \frac{1}{2} \frac{\Gamma(5/6)}{\Gamma(11/6)} \left(\frac{z}{L_0^2}\right) - \frac{1}{2} \Gamma(-11/6)\left(\frac{z}{L_0^2}\right)^{11/6} \Phi\,(-11/6|1|-\beta^2/4z)} \qquad \text{(D-43)}$$

$$\left|\frac{z}{L_0^2}\right| \ll 1 \quad .$$

Collecting the various partial results developed above, we find:

$$B_A(\vec{\rho}) \to 2\pi^{\frac{1}{2}} <n^2> k^3 L_0^3 \frac{\Gamma(11/6)}{(1/3)} \lim_{\gamma \to 0^+} *$$

$$\Big[\text{Im } K(\gamma + i \frac{\rho}{2k} \sin\theta) - \frac{\rho}{2k} \sin\theta \text{ Re } I(\gamma + i \frac{\rho}{2k} \sin\theta)$$

$$- \text{Im } K(\gamma + i \frac{L}{k}) + \frac{L}{k} \text{ Re } I(\gamma + i \frac{\rho}{2k} \sin\theta)\Big]$$

$$\rho << \frac{4\pi L_0^2}{\lambda \sin\theta} \quad ; \quad \sqrt{\lambda L} << L_0 \quad .$$

For normalization purposes, let us consider:

$$B_A(\vec{0}) \to 2\pi^{\frac{1}{2}} <n^2> k^3 L_0^3 \frac{\Gamma(11/6)}{\Gamma(1/3)} \lim_{\gamma \to 0^+} \left[\frac{L}{k} I(\gamma) - \text{Im } K(\gamma + i \frac{L}{k})\right] .$$

Upon taking $\beta = 0$ in both $I(z)$ and $K(z)$ we obtain:

$$I(\gamma) = \frac{1}{2L_0^2} \frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{1}{2L_0^2} \Gamma(-5/6) \left(\frac{\gamma}{L_0^2}\right)^{5/6}$$

$$K(\gamma + i \frac{L}{k}) = \frac{1}{2} \frac{\Gamma(5/6)}{(11/6)} \frac{\gamma + iL/k}{L_0^2} - \frac{1}{2} \Gamma(-11/6) \left(\frac{\gamma + i L/k}{L_0^2}\right)^{11/6}$$

having employed

$$\Phi(a|c|0) = 1 \quad .$$

Therefore,

$$B_A(\vec{0}) \to 2\pi^{\frac{1}{2}} < n^2 > k^3 L_0^3 \frac{\Gamma(11/6)}{(1/3)} 2^{5/6} \Gamma(-11/6) \sin\left(\frac{11\pi}{12}\right) *$$

$$k^{-11/6} (2L_0^2)^{-11/6} L^{11/6} \tag{D-44}$$

and introducing the normalized correlation function

$$b_A(\vec{\rho}) \equiv B_A(\vec{\rho})/B_A(\vec{0})$$

we shall have:

$$\boxed{\begin{aligned} b_A(\vec{\rho}) \to \delta \left(\frac{L}{k}\right)^{-11/6} \lim_{\gamma \to 0^+} & * \\ & \left[\mathrm{Im}\, K(\gamma + i \frac{\rho}{2k} \sin\theta) - \frac{\rho}{2k} \sin\theta \,\mathrm{Re}\, I(\gamma + i\frac{\rho}{2k} \sin\theta)\right. \\ & \left. - \mathrm{Im}\, K(\gamma + i\frac{L}{k}) + \frac{L}{k} \mathrm{Re}\, I(\gamma + i \frac{\rho}{2k} \sin\theta)\right] \end{aligned}} \tag{D-45}$$

$$\rho << \frac{4\pi^2 L_0^2}{\lambda \sin\theta} \quad ; \qquad \sqrt{\lambda L} << L_0^2$$

for

$$\delta \equiv (2L_0^2)^{11/6} \left[2^{5/6} \Gamma(-11/6) \sin\left(\frac{11}{12}\pi\right)\right]^{-1} .$$

The result of equation D-44 is quite important in itself, because

$$B_A(\vec{0}) = \sigma_A^2$$

where σ_A stands for the variance of the logorithmic amplitude distribution. In fact, we find:

$$\sigma_A^2 \sim \langle n^2 \rangle k^{7/6} L^{11/6} \quad .$$

The mean square refractive index fluctuation $\langle n^2 \rangle$ can be eliminated in favor of the experimentally measurable C_n^2 that appears in the Kolmogorov "two-thirds" law,

$$D_n(\rho) = C_n^2 \, \rho^{2/3} \quad .$$

Indeed, using the power series expansion of the $K_{1/3}(x)$ Bessel function and retaining only the first two terms for $\rho/L_0 \ll 1$, the Karman structure function becomes:

$$D_n(\rho) \rightarrow 2 \langle n^2 \rangle \frac{\Gamma(2/3)}{\Gamma(4/3)} \left(\frac{\rho}{2L_0^2}\right)^{2/3} \quad .$$

Upon comparison with the Kolmogorov equation,

$$\langle n^2 \rangle = C_n^2 \frac{\Gamma(4/3)}{2\Gamma(2/3)} (2L_0)^{2/3}$$

and therefore:

$$\sigma_A^2 = .31 \; C_n^2 \; k^{7/6} \; L^{11/6}$$

in perfect agreement with the result of an earlier analysis performed by Tatarski.[1]

We shall now make use of the asymptotic behavior of the Kummer function,

$$\Phi(a|c|x) \to \frac{\Gamma(c)}{\Gamma(c-a)} (e^{i\epsilon\pi}/x)^a \sum_{n=0}^{M} \frac{(a)_n (a-c+1)_n}{n!} (-x)^{-n}$$

$$+ \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \sum_{n=0}^{N} \frac{(c-a)_n (1-a)_n}{n!} x^{-n}$$

with

$$(a)_0 = 1 \quad ; \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad , \quad n \geq 1$$

$$\epsilon = 1 \text{ if } \operatorname{Im} x > 0 \quad , \qquad \epsilon = -1 \text{ if } \operatorname{Im} x < 0 \quad ;$$

$$-\pi < \arg x < \pi$$

as well as of its behavior near the origin to extract from equation D-45 information concerning the normalized correlation function $b_A(\vec{\rho})$. One can thus show, after some lengthy algebra that as long as θ does not exceed $\tan^{-1}(\sqrt{L/\lambda})$,

$$b_A(\vec{\rho}) \to 1 - 11/6 \left(\frac{\rho}{2L} \sin \theta\right)^{5/6} \quad ;$$

$$\text{for} \qquad \rho \ll \lambda \frac{\tan \theta}{\cos \theta}$$

$$b_A(\vec{\rho}) \to 1 - \frac{11/6}{2^{5/3}\Gamma(11/6) \sin (11/12\,\pi)} \left(\frac{k}{L}\right)^{5/6} (\rho \cos \theta)^{5/3} \quad ;$$

$$\text{for} \quad \lambda \frac{\tan \theta}{\cos \theta} \ll \rho \ll \frac{\sqrt{\lambda L}}{\cos \theta} \tag{D-46}$$

$$b_A(\vec{\rho}) \to - \frac{2^{7/3}\,\Gamma^2(7/6)}{11\Gamma(11/6)\Gamma^2(-11/6) \sin (11/12\,\pi)} \left(\frac{L}{k}\right)^{7/6} (\rho \cos \theta)^{-7/3} \quad ;$$

$$\text{for} \qquad \rho \gg \frac{\sqrt{\lambda L}}{\cos \theta}$$

It follows from equation D-46 that the normalized correlation function $b_A(\vec{\rho})$ starts off at 1, decreases as one minus the five-thirds power of the normalized distance $\rho \cos\theta / \sqrt{\lambda L}$ until it crosses the abscisa somewhere around $\sqrt{\lambda L}/\cos\theta$, and then approaches zero asymptotically from below.

An exact representation of the correlation function $b_A(\vec{\rho})$ is given in figure 4, for $\theta = 10^0$.

The correlation length of the logorithmic amplitude distribution is thus of order

$$\rho_{corr} \sim \frac{\sqrt{\lambda L}}{\cos \theta} \quad .$$

For $\theta = 0$, this result is in perfect agreement with the Tatarski calculation in the $\sqrt{\lambda L} << L_0$ regime, and is therefore the natural generalization thereof away from transversal correlation. As θ nears $\pi/2$ the result of equation D-45 is no longer accurate. In fact, for $\theta = \theta_0$ the solution of

$$\frac{4\pi\, L_0^2}{\lambda \sin\theta} = \frac{\sqrt{\lambda L}}{\cos\theta} \quad ,$$

the correlation length is of order $4\pi\, L_0^2/\lambda \sin\theta$ and hence values of $\rho \gtrsim 4\pi\, L_0^2/\lambda \sin\theta$ become relevant. It can be argued, however, that for values of $\theta \geq \theta_0$ the correlation length becomes

$$\rho_{corr} \sim \frac{4\pi\, L_0^2}{\lambda \sin\theta} \quad . \tag{D-47}$$

Indeed, introducing

$$\mu \equiv \frac{\lambda \rho \sin\theta}{4\pi\, L_0^2}$$

we can still use equation D-45 to describe the behavior of the normalized correlation function for $\mu << 1$. When $\mu >> 1$, we can return to equation D-41 and use the asymptotic behavior of the Tricomi function,

$$\Psi\,(a|c|x) \rightarrow \sum_{n=0}^{N} (-1)^n \; \frac{(a)_n\,(a-c+1)_n}{n!} \; x^{-a-n}$$

$$-\frac{3}{2}\pi < \arg x < \frac{3}{2}\pi$$

to have:

$$I(z) \rightarrow \frac{1}{2z}\, e^{-\beta^2/4z} \quad .$$

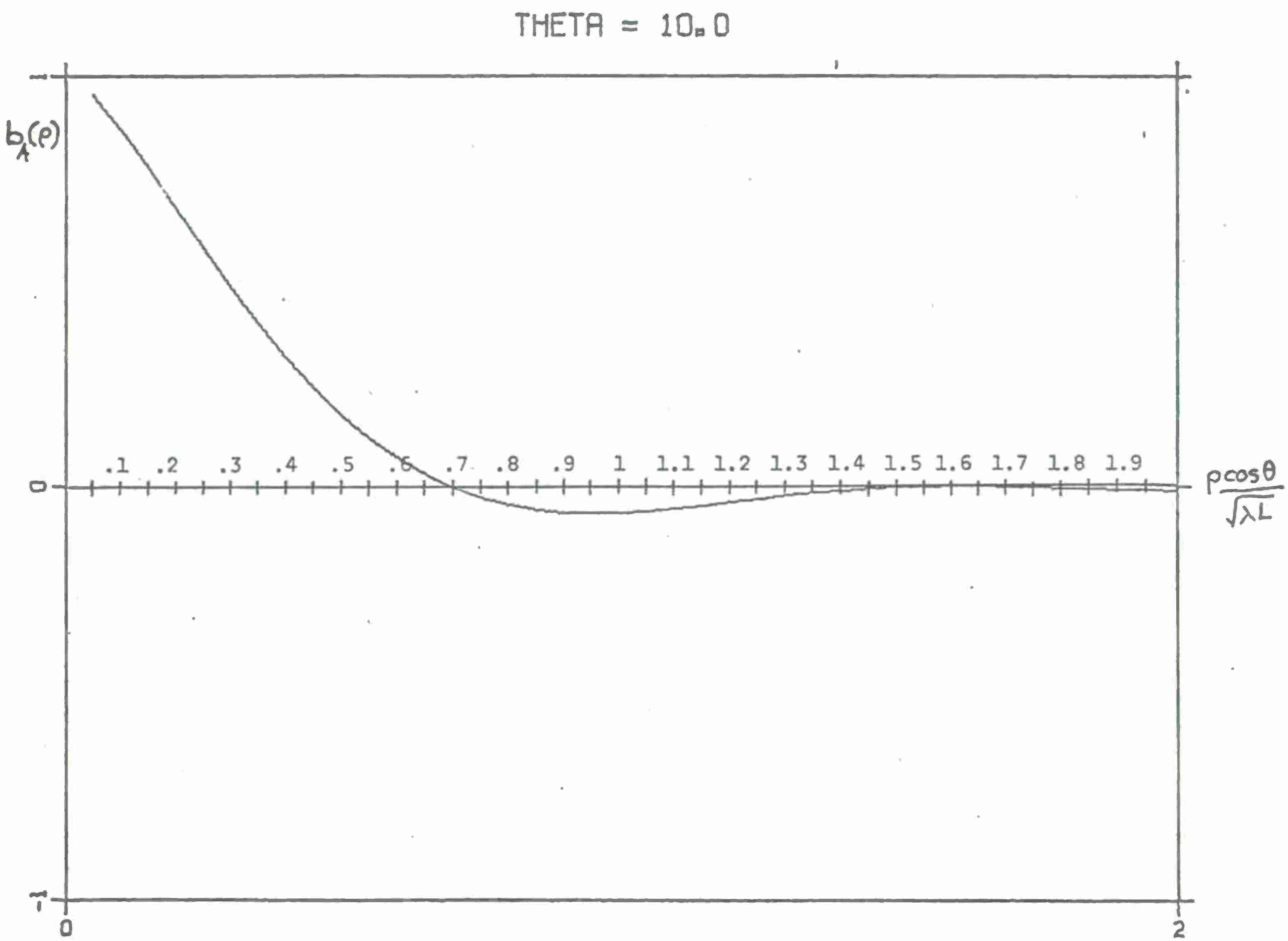

FIG. 4: THE NORMALIZED AMPLITUDE CORRELATION FUNCTION

Correspondingly,

$$K(z) \to - \mathrm{Ei}\,(-\beta^2/4z)$$

providing for the correlation function a highly oscillatory behavior in the $\mu >> 1$ region. The change of regime manifested by $b_A(\vec{\rho})$ as μ goes through 1, indicates that the correlation length corresponds to values of μ near 1 and hence that equation D-47 holds. To obtain the correct shape of the correlation function $b_A(\vec{\rho})$ as μ passes 1 would require summing up the series in equation D-41 without any approximation on the Tricomi functions $\Psi\,(n+1\,|n+1/6\,|z/L_0^2)$. We shall restrain from attempting to do so here.

For $\theta = \pi/2$ our result corresponds exactly to the one obtained in 1961 by Chernov.[10]

5. COMPARISON WITH EXPERIMENTAL DATA

The value of the variance,

$$\sigma_A^2 = .31\, C_n^2\, k^{7/6}\, L^{11/6} \quad ; \quad \sqrt{\lambda L} << L_0 \quad ,$$

and the correlation length

$$\rho_{corr} \sim \begin{cases} \dfrac{\sqrt{\lambda L}}{\cos\,\theta}\;, & 0 \le \theta < \theta_0 \\ & \qquad\qquad ; \; \sqrt{\lambda L} << L_0 \\ \dfrac{4\pi\, L_0^2}{\lambda\, \sin\,\theta}\;, & \theta_0 < \theta \le \pi/2 \end{cases}$$

for the logarithmic amplitude fluctuation field of a monocromatic sound plane wave propagating in a turbulent ocean, are our main result. It is to be observed that they depend explicitly on the wave number k of the propagating sound and on the distance L between the source and the receiver, and implicitly through C_n^2 on the environmental conditions prevailing in the region of propagation.

We shall conclude this analysis by providing some information concerning the comparison of our result at $\theta = 0$ with adequately taken experimental data. The experiments we have in mind, were conducted in 1961, and then again in 1962, 1964, from two scientific research vessels belonging to the Academy of Sciences of the USSR, the Sergei Vavilov and the Peter Lebedeev.[3] Periodically repeated pulsed signals with a tone frequency carrier f were recorded at a definite distance from the transmitter. The depth H of the transmitter and receiver, the distance L between them, the pulse duration $\Delta\tau$, and their period of repetition were chosen so as to permit time separation of the direct and surface-reflected signals. As a rule, simultaneously with the recording of the acoustic signals, the fluctuations of the velocity of sound were measured at the depth of the receiver. The receiver was designed to operate in conjunction with a device permitting the fluctuations of the refractive index to be measured to 10^{-6} by the microphasometric method. This simultaneous recording procedure provided the opportunity of analyzing the results to include the variation of the statistical characteristics of the refractive index fluctuations in the course of the experiment. The statistical processing of the recorded fluctuations of the index of refraction and fluctuations of the acoustic pressure level of the direct pulses consisted in calculating the time structure functions $D(\tau)$ according to,

$$D(\tau) = \left[\sum_{i=1}^{N-m} \left(z_{i+m} - Z_i\right)^2\right] / (N-m)$$

where z_i is the value of the ith sampling of the refractive index or logarithmic amplitude, N is the total numbers of samplings in a recorded segment of duration T, $\Delta\tau = T/N - 1$ is the partition subinterval, and $m = 1, 2, \ldots$ is the number of partition subintervals $\Delta\tau$ in the interval τ. The partition subinterval $\Delta\tau$ for the recordings of the acoustic pulses was equal to their period of repetition ΔT; for the refraction fluctuation recordings it was equal to 0.05 second. The number N was between the limits 600 and 1,200.

The transition from the time scale τ to the space scale ρ was performed according to the formula,

$$\rho = v\tau \quad .$$

The structure functions of the refractive index fluctuations $D_n(v\tau)$ and the amplitude fluctuations $D_A(v\tau)$ were used to ascertain the correlation radii and mean-square fluctuations of the refractive index and of the logorithmic amplitude. The principal characteristics of the experiments and the results of their processing are shown in table 1.

Figure 5 shows the normalized structure functions $D_n(\rho)/2<n^2>$ for fluctuations of the refractive index for the recordings obtained in the 1962 tests. It indicates that for values of $\rho < L_0$ these functions on the average fairly well satisfy the "two-thirds" law of Komogorov and Obukhov. In the 1964 experiments direct measurement of the inhomogeneities were not made, but there existed good evidence that the "two-thirds" law was valid up to scales of at least 60 miles.

TABLE 1: EXPERIMENTAL CHARACTERISTICS AND RESULTS

TIME AND PLACE OF EXPERIMENT	H, m	ν, kHz	Δt, msec.	ΔT, sec.	L, m	$\sqrt{\lambda L}$, m	$\sigma_A \times 10^2$	ρ_{corr}, m	$\sqrt{\langle n^2 \rangle} \times 10^3$	L_0, m
					200	3.5	9.5	.36	13.7	.63
1962 May, Northwest Atlantic	40	25	1	.4	665	6.3	26.4	.46	25.4	.28
					900	7.4	18.7	.11	29.0	.11
					1150	8.3	14.1	1.16	4.7	1.94
1962 June, Northeast Atlantic	35	25	1	.4	910	7.5	7.1	1.58	2.7	1.98
					600	6.0	10.0	2.22	5.4	2.66
					480	5.4	15.8	.80	11.0	.80
1962 July, Sea of Norway	20	25	1	.4	240	3.8	17.3	.76	17.8	.76
					160	3.1	17.3	.60	25.2	.61
					1490	23.6	2.0	27.4		
1964 March, Sargasso Sea	250	4	10	2.0	4100	39.0	9.6	38.2		
					8730	57.0	17.0	54.1		
					250	11.0	2.2	12.1		
					700	18.5	5.8	16.9		
1964 May, Sea of Norway	150	3	10	1.0	1260	25.0	9.0	31.6		
					2850	37.8	17.8	37.8		
					5220	51.0	25.0	53.0		

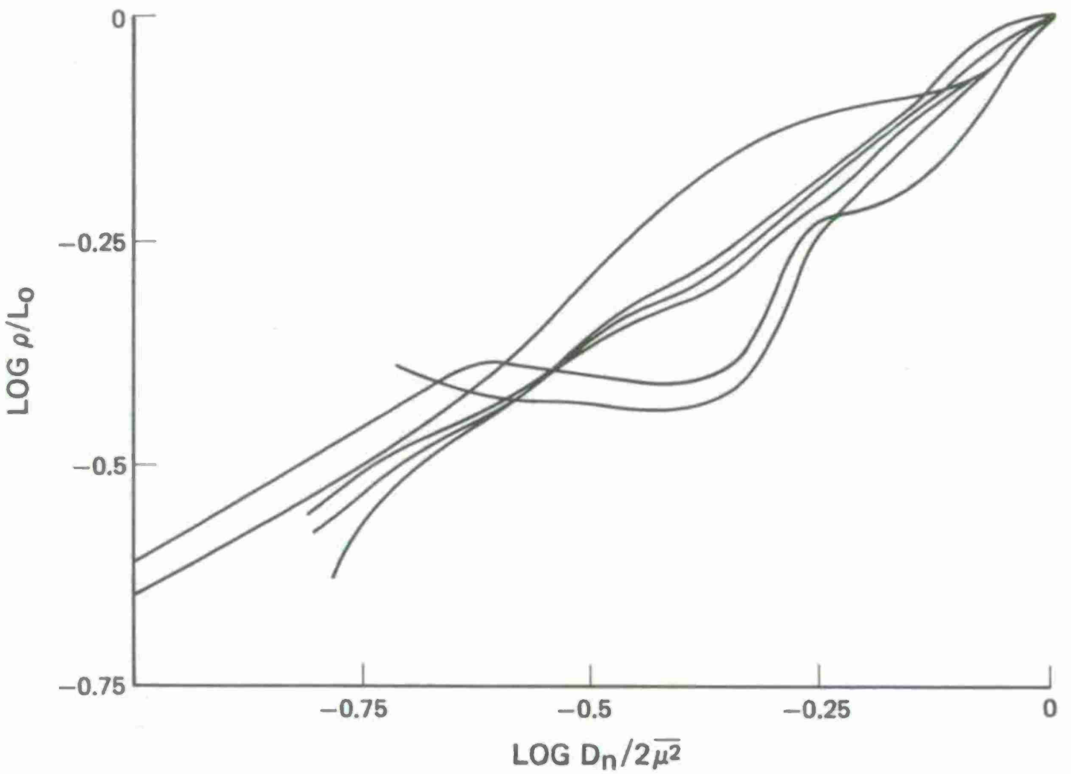

FIG. 5: THE TEMPERATURE STRUCTURE FUNCTION

The normalized structure functions $D_A(\rho)/2\sigma_A^2$ obtained from processing of the recordings of a sequence of pulses at various distances are constructed as a function of $\rho/\sqrt{\lambda L}$ in figure 6. Notice that the curves for the structure functions roughly coincide at all distances and intersect the saturation level between values of the normalized distance from .9 to 1.3. This indicates that the correlation distance is of order $\sqrt{\lambda L}$ when $\sqrt{\lambda L} << L_0$.

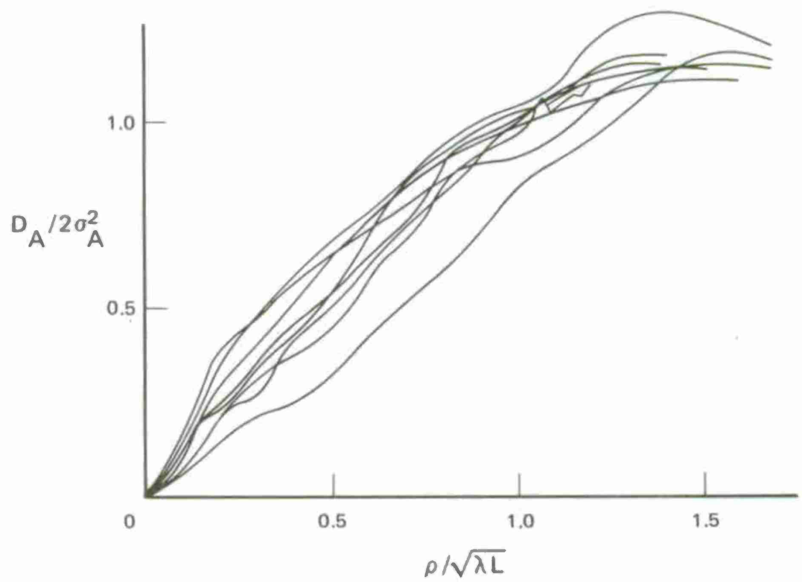

FIG. 6: THE AMPLITUDE STRUCTURE FUNCTION

Finally the dependence of $\left[\sigma_A^2/.31\ C_n^2\ K^{7/6}\right]^{\frac{1}{2}}$ on L is depicted in figure 7. The line drawn through the experimental points by the method of the least squares almost perfectly coincides with the theoretical dependence of $L^{11/12}$.

The foregoing discussion concerning the results of experiments on the fluctuations of the sound level in an ocean medium containing inhomogeneities of the velocity of sound shows that a random field of the refactive index may be described to a satisfactory approximation by means of structure functions satisfying the Komogorov-Obukhov "two-thirds" law with external inhomogeneity scales L_0 ranging from several tens of centimeters to several tens of meters, and the experimental dependences of the mean-square and radius of correlation of the acoustic pressure level fluctuations on the distance are in good agreement with the theoretical predictions.

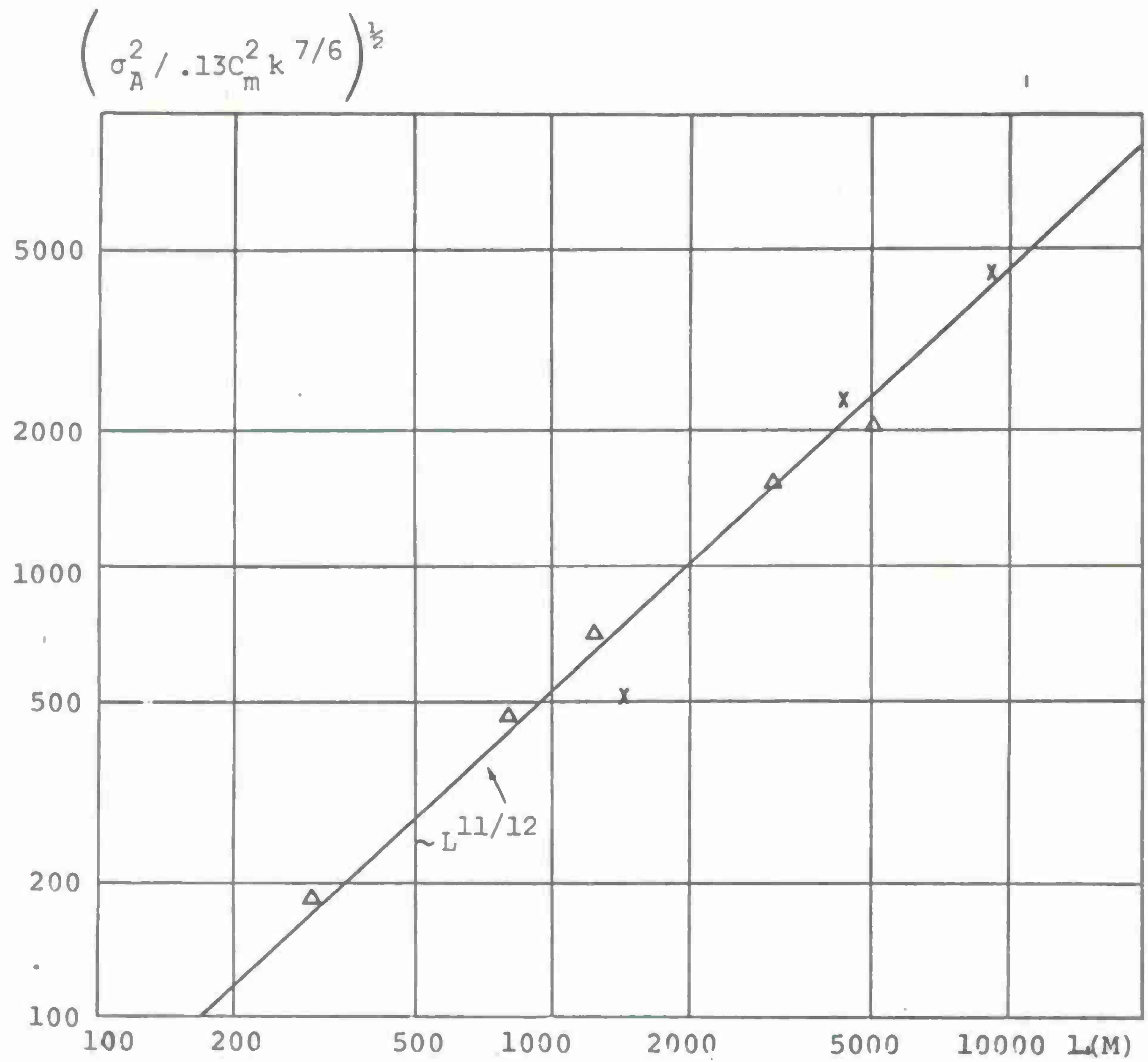

FIG. 7: RANGE DEPENDENCE OF CORRELATION COEFFICIENT

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